MATHEMATICS REVIEW

for the

FUNDAMENTALS OF ENGINEERING EXAMINATION

Presented By

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NATIONAL COUNCIL OF EXAMINERS FOR ENGINEERING AND SURVEYING

FUNDAMENTALS OF ENGINEERING

(FE)

DISCIPLINE SPECIFIC

REFERENCE HANDBOOK

Second Printing, September 1996

National Council of Examiners for Engineering and Surveying (NCEES)
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www.ncees.org

MATHEMATICS

STRAIGHT LINE

The general form of the equation is

$$Ax + By + C = 0$$

The standard form of the equation is

$$y = mx + b,$$

which is also known as the *slope-intercept* form.

The *point-slope* form is $\quad y - y_1 = m(x - x_1)$

Given two points: slope, $\quad m = (y_2 - y_1)/(x_2 - x_1)$

The angle between lines with slopes m_1 and m_2 is

$$\alpha = \arctan\,[(m_2 - m_1)/(1 + m_2 \cdot m_1)]$$

Two lines are perpendicular if $\quad m_1 = -1/m_2$

The distance between two points is

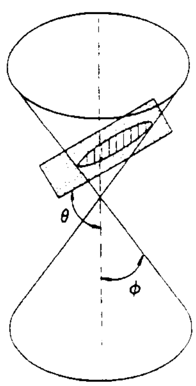
$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

QUADRATIC EQUATION

$$ax^2 + bx + c = 0$$

$$\text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

CONIC SECTIONS



$$e = \text{eccentricity} = \cos\theta/(\cos\phi)$$

[Note: X' and Y', in the following cases, are translated axes.]

Case 1. Parabola $\quad e = 1$:

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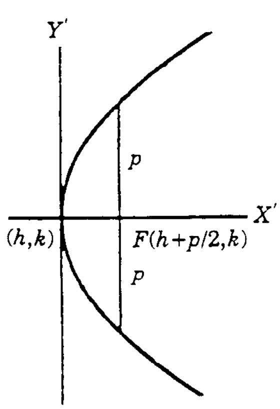

$$(y - k)^2 = 2p(x - h);\ \text{Center at } (h,k)$$

is the standard form of the equation. Then, when $h = k = 0$,

Focus: $(p/2, 0)$; Directrix: $x = -p/2$

Case 2. Ellipse $\quad e < 1$:

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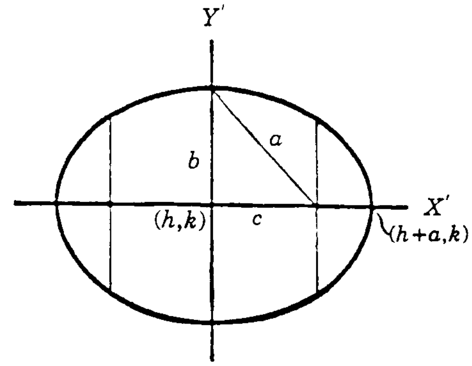

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1;\ \text{Center at } (h,k)$$

is the standard form of the equation. When $h = k = 0$,

Eccentricity: $e = \sqrt{1 - (b^2/a^2)} = c/a$

$b = a\sqrt{1 - e^2}$;

Focus: $(\pm ae, 0)$; Directrix: $x = \pm a/e$

Case 3. Hyperbola $\quad e > 1$:

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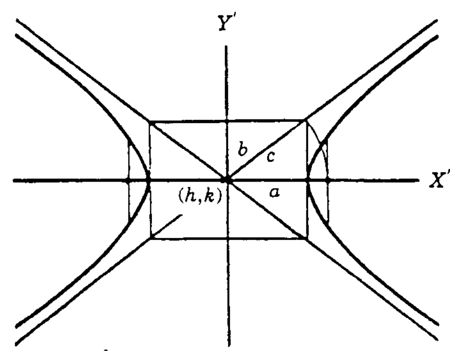

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1;\ \text{Center at } (h,k)$$

is the standard form of the equation. When $h = k = 0$,

Eccentricity: $e = \sqrt{1 + (b^2/a^2)} = c/a$

$b = a\sqrt{e^2 - 1}$;

Focus: $(\pm ae, 0)$; Directrix: $x = \pm a/e$

Case 4. Circle $\quad e = 0$:

$$(x - h)^2 + (y - k)^2 = r^2;\ \text{Center at } (h,k)$$

is the general form of the equation with radius

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

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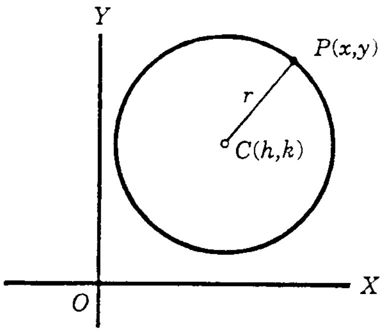

Length of the tangent from a point. Using the general form of the equation of a circle, the length of the tangent is found from

$$t^2 = (x' - h)^2 + (y' - k)^2 - r^2$$

by substituting the coordinates of a point $P(x', y')$ and the coordinates of the center of the circle into the equation and computing.

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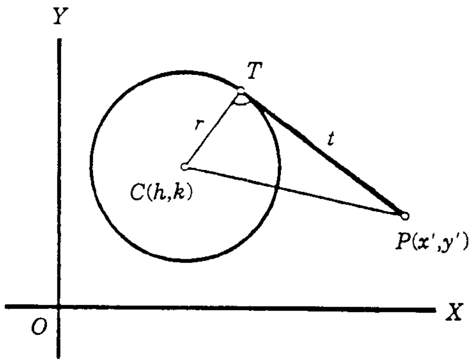


Conic Section Equation

The general form of the conic section equation is

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

where not both A and C are zero.
If $B^2 - AC < 0$, an *ellipse* is defined.
If $B^2 - AC > 0$, a *hyperbola* is defined.
If $B^2 - AC = 0$, the conic is a *parabola*.
If $A = C$ and $B = 0$, a *circle* is defined.
If $A = B = C = 0$, a *straight line* is defined.

$$x^2 + y^2 + 2ax + 2by + c = 0$$

is the normal form of the conic section equation, if that conic section has a principal axis parallel to a coordinate axis.

$$h = -a; \quad k = -b$$

$$r = \sqrt{a^2 + b^2 - c}$$

If $a^2 + b^2 - c$ is positive, a *circle*, center $(-a, -b)$.
If $a^2 + b^2 - c$ equals zero, a *point* at $(-a, -b)$.
If $a^2 + b^2 - c$ is negative, locus is *imaginary*.

QUADRIC SURFACE (SPHERE)

The general form of the equation is

$$(x - h)^2 + (y - k)^2 + (z - m)^2 = r^2$$

with center at (h, k, m).

In a three-dimensional space, the distance between two points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

LOGARITHMS

The logarithm of x to the Base b is defined by

$$\log_b(x) = c, \text{ where } \quad b^c = x$$

Special definitions for $b = e$ or $b = 10$ are:

$$\ln x, \text{ Base } = e$$

$$\log x, \text{ Base } = 10$$

To change from one Base to another:

$$\log_b x = (\log_a x)/(\log_a b)$$

e.g., $\ln x = (\log_{10} x)/(\log_{10} e) = 2.302585 \, (\log_{10} x)$

Identities $\log_b b^n = n$

$$\log x^c = c \log x; \quad x^c = \text{antilog}\,(c \log x)$$

$$\log xy = \log x + \log y$$

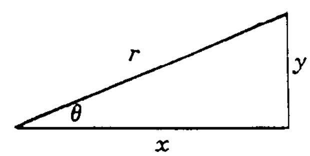
$$\log_b b = 1; \quad \log 1 = 0$$

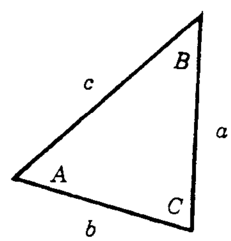
$$\log x/y = \log x - \log y$$

TRIGONOMETRY

Trigonometric functions are defined using a right triangle.

$\sin\theta = y/r$, $\cos\theta = x/r$
$\tan\theta = y/x$, $\cot\theta = x/y$
$\csc\theta = r/y$, $\sec\theta = r/x$





Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Identities

$\csc\theta = 1/\sin\theta$; $\quad \tan\theta = \sin\theta/\cos\theta$

$\sec\theta = 1/\cos\theta$; $\quad \cot\theta = 1/\tan\theta$

$\sin^2\theta + \cos^2\theta = 1$; $\quad \tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$

$\tan 2\alpha = (2\tan\alpha)/(1 - \tan^2\alpha)$

$\cot 2\alpha = (\cot^2\alpha - 1)/(2\cot\alpha)$

$\tan(\alpha + \beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$

$\cot(\alpha + \beta) = (\cot\alpha\cot\beta - 1)/(\cot\alpha + \cot\beta)$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

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$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\tan(\alpha-\beta) = (\tan\alpha - \tan\beta)/(1+\tan\alpha\tan\beta)$$
$$\cot(\alpha-\beta) = (\cot\alpha\cot\beta + 1)/(\cot\beta - \cot\alpha)$$
$$\sin(\alpha/2) = \pm\sqrt{(1-\cos\alpha)/2}$$
$$\cos(\alpha/2) = \pm\sqrt{(1+\cos\alpha)/2}$$
$$\tan(\alpha/2) = \pm\sqrt{(1-\cos\alpha)/(1+\cos\alpha)}$$
$$\cot(\alpha/2) = \pm\sqrt{(1+\cos\alpha)/(1-\cos\alpha)}$$
$$\sin\alpha\sin\beta = (1/2)[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$
$$\cos\alpha\cos\beta = (1/2)[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$
$$\sin\alpha\cos\beta = (1/2)[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$
$$\sin\alpha + \sin\beta = 2\sin(1/2)(\alpha+\beta)\cos(1/2)(\alpha-\beta)$$
$$\sin\alpha - \sin\beta = 2\cos(1/2)(\alpha+\beta)\sin(1/2)(\alpha-\beta)$$
$$\cos\alpha + \cos\beta = 2\cos(1/2)(\alpha+\beta)\cos(1/2)(\alpha-\beta)$$
$$\cos\alpha - \cos\beta = -2\sin(1/2)(\alpha+\beta)\sin(1/2)(\alpha-\beta)$$

COMPLEX NUMBERS

Definition $i = \sqrt{-1}$

$$(a+ib)+(c+id) = (a+c)+i(b+d)$$
$$(a+ib)-(c+id) = (a-c)+i(b-d)$$
$$(a+ib)(c+id) = (ac-bd)+i(ad+bc)$$
$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$
$$(a+ib)+(a-ib) = 2a$$
$$(a+ib)-(a-ib) = 2ib$$
$$(a+ib)(a-ib) = a^2+b^2$$

Polar Coordinates

$$x = r\cos\theta;\quad y = r\sin\theta;\quad \theta = \arctan(y/x)$$
$$r = |x+iy| = \sqrt{x^2+y^2}$$
$$x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$
$$[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$
$$(x+iy)^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$
$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

Euler's Identity
$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$
$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2},\quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Roots

If k is any positive integer, any complex number (other than zero) has k distinct roots. The k roots of $r(\cos\theta + i\sin\theta)$ can be found by substituting successively $n = 0, 1, 2, \ldots, (k-1)$ in the formula

$$w = \sqrt[k]{r}\left[\cos\left(\frac{\theta}{k} + n\,\frac{360°}{k}\right) + i\sin\left(\frac{\theta}{k} + n\,\frac{360°}{k}\right)\right]$$

MATRICES

A matrix is an ordered rectangular array of numbers with m rows and n columns. The element a_{ij} refers to row i and column j.

Multiplication

If $A = (a_{ik})$ is an $m \times n$ matrix and $B = (b_{kj})$ is an $n \times s$ matrix, the matrix product AB is an $m \times s$ matrix

$$C = (c_{ij}) = \left(\sum_{l=1}^{n} a_{il}b_{lj}\right)$$

where n is the common integer representing the number of columns of A and the number of rows of B (l and $k = 1, 2, \ldots, n$).

Addition

If $A = (a_{ij})$ and $B = (b_{ij})$ are two matrices of the same size $m \times n$, the sum $A + B$ is the $m \times n$ matrix $C = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$.

Identity

The matrix $I = (a_{ij})$ is a square $n \times n$ identity matrix where $a_{ii} = 1$ for $i = 1, 2, \ldots, n$ and $a_{ij} = 0$ for $i \neq j$.

Transpose

The matrix B is the transpose of the matrix A if each entry b_{ji} in B is the same as the entry a_{ij} in A and conversely. In equation form, the transpose is

$$B = A^T$$

Inverse

The inverse B of a square $n \times n$ matrix A is

$$B = A^{-1} = \frac{\mathrm{adj}(A)}{|A|},\quad \text{where}$$

adj(A) = adjoint of A (obtained by replacing A^T elements with their cofactors, see **DETERMINANTS**) and

$|A|$ = determinant of A.

DETERMINANTS

A *determinant of order* n consists of n^2 numbers, called the *elements* of the determinant, arranged in n rows and n columns and enclosed by two vertical lines. In any determinant, the *minor* of a given element is the determinant that remains after all of the elements are struck out that lie in the same row and in the same column as the given element. Consider an element which lies in the hth column and the kth row. The *cofactor* of this element is the value of the minor of the element (if $h + k$ is *even*), and it is the negative of the value of the minor of the element (if $h + k$ is *odd*).

If n is greater than 1, the *value* of a determinant of order n is the sum of the n products formed by multiplying each element of some specified row (or column) by its cofactor. This sum is called the *expansion of the determinant* [according to the elements of the specified row (or column)].

$$\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} \overset{14}{2 \times 7} + \overset{24}{3 \times 8} \\ \overset{35}{5 \times 7} + \overset{48}{6 \times 8} \end{bmatrix} = \begin{bmatrix} 38 \\ 83 \end{bmatrix}$$

$$[I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A] \qquad [A]^{-1}$$

$$[A][A]^{-1} = [I]$$

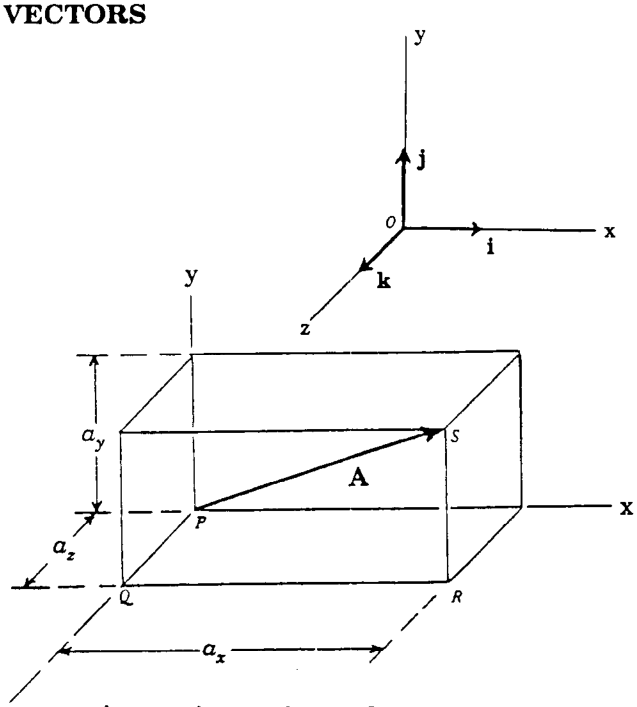
For a second-order determinant:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

For a third-order determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

VECTORS



$$\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

Addition and *subtraction*:

$$\mathbf{A} + \mathbf{B} = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k}$$

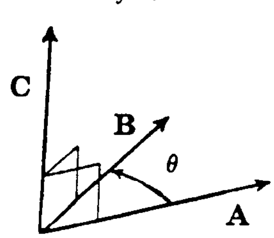
$$\mathbf{A} - \mathbf{B} = (a_x - b_x)\mathbf{i} + (a_y - b_y)\mathbf{j} + (a_z - b_z)\mathbf{k}$$

The *dot product* is a *scalar product* and represents the projection of $\mathbf{B}$ onto $\mathbf{A}$ times $\mathbf{A}$. It is given by

$$\mathbf{A}\cdot\mathbf{B} = a_xb_x + a_yb_y + a_zb_z = |\mathbf{A}|\,|\mathbf{B}|\cos\theta = \mathbf{B}\cdot\mathbf{A}$$

The *cross product* is a *vector product* of magnitude $|\mathbf{B}|\,|\mathbf{A}|\sin\theta$ which is perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$. The product is

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = -\mathbf{B}\times\mathbf{A}$$



The sense of $\mathbf{A}\times\mathbf{B}$ is determined by the right-hand rule.

$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\,\mathbf{n}\sin\theta, \text{ where}$$

$\mathbf{n}$ = unit vector perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$.

Identities

$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$; $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$

$\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2$

$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$

$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$

If $\mathbf{A}\cdot\mathbf{B} = 0$, then either $\mathbf{A} = 0$, $\mathbf{B} = 0$, or $\mathbf{A}$ is perpendicular to $\mathbf{B}$.

$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$

$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$

$(\mathbf{B}+\mathbf{C})\times\mathbf{A} = \mathbf{B}\times\mathbf{A} + \mathbf{C}\times\mathbf{A}$

$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$

$\mathbf{i}\times\mathbf{j} = \mathbf{k} = -\mathbf{j}\times\mathbf{i}$; $\mathbf{j}\times\mathbf{k} = \mathbf{i} = -\mathbf{k}\times\mathbf{j}$

$\mathbf{k}\times\mathbf{i} = \mathbf{j} = -\mathbf{i}\times\mathbf{k}$

If $\mathbf{A}\times\mathbf{B} = 0$, then either $\mathbf{A} = 0$, $\mathbf{B} = 0$, or $\mathbf{A}$ is parallel to $\mathbf{B}$.

PROGRESSIONS AND SERIES

Arithmetic Progression

To determine whether a given finite sequence of numbers is an arithmetic progression, subtract each number from the following number. If the differences are equal, the series is arithmetic.

1. The first term is a.
2. The common difference is d.
3. The number of terms is n.
4. The last or nth term is l.
5. The sum of n terms is S.

$$l = a + (n-1)d$$

$$S = n(a+l)/2 = n[2a + (n-1)d]/2$$

Geometric Progression

To determine whether a given finite sequence is a geometric progression, divide each number after the first by the preceding number. If the quotients are equal, the series is geometric.

1. The first term is a.
2. The common ratio is r.
3. The number of terms is n.
4. The last or nth term is l.
5. The sum of n terms is S.

$$l = ar^{n-1}$$

$$S = a(1-r^n)/(1-r); \quad r \neq 1$$

$$S = (a-rl)/(1-r); \quad r \neq 1$$

$$\lim_{n\to\infty} S_n = a/(1-r); \quad r < 1$$

A G.P. converges if $|r| < 1$ and it diverges if $|r| \geq 1$.

Properties of Series

$$\sum_{i=1}^{n} c = nc; \quad c = \text{constant}$$

$$\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} (x_i + y_i - z_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i$$

$$\sum_{x=1}^{n} x = (n + n^2)/2$$

1. A power series in x, or in $x-a$, which is convergent in the interval $-1 < x < 1$ (or $-1 < x-a < 1$), defines a function of x which is continuous for all values of x within the interval and is said to represent the function in that interval.
2. A power series may be differentiated term by term, and the resulting series has the same interval of convergence as the original series (except possibly at the end points of the interval).
3. A power series may be integrated term by term provided the limits of integration are within the interval of convergence of the series.
4. Two power series may be added, subtracted, or multiplied, and the resulting series in each case is convergent, at least, in the interval common to the two series.
5. Using the process of long division (as for polynomials), two power series may be divided one by the other.

Taylor's Series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

is called *Taylor's series*, and the function $f(x)$ is said to be expanded about the point a in a Taylor's series.

If $a = 0$, the Taylor's series equation becomes a *Maclaurin's series.*

PROBABILITY AND STATISTICS

Permutations and Combinations

A *permutation* is a particular sequence of a given set of objects. A *combination* is the set itself without reference to order.

1. The number of different *permutations* of n distinct objects *taken* r *at a time* is

$$P(n,r) = \frac{n!}{(n-r)!}$$

2. The number of different *combinations* of n distinct objects *taken* r *at a time* is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

3. The number of different *permutations* of n objects *taken* n *at a time*, given that n_i are of type i, where $i = 1, 2, \ldots, k$ and $\Sigma\, n_i = n$, is

$$P(n;n_1,n_2,\ldots,n_k) = \frac{n!}{n_1!n_2!\ldots n_k!}$$

Laws of Probability

***PROPERTY* 1** (General Character of Probability) The probability $P(E)$ of an event E is a real number in the range of 0 to 1. The probability of an impossible event is 0 and that of an event certain to occur is 1.

***PROPERTY* 2** (Law of Total Probability)

$$P(A + B) = P(A) + P(B) - P(A,B), \quad \text{where}$$

$P(A + B)$ = the probability that either A or B occur alone or that both occur together,
$P(A)$ = the probability that A occurs,
$P(B)$ = the probability that B occurs, and
$P(A,B)$ = the probability that both A and B occur simultaneously.

***PROPERTY* 3** (Law of Compound or Joint Probability) If neither $P(A)$ nor $P(B)$ is zero,

$$P(A,B) = P(A)P(B\backslash A) = P(B)P(A\backslash B),$$

where

$P(B\backslash A)$ = the probability that B occurs given the fact that A has occurred, and
$P(A\backslash B)$ = the probability that A occurs given the fact that B has occurred.

If either $P(A)$ or $P(B)$ is zero, then

$$P(A,B) = 0$$

Probability Functions

A random variable x has a probability associated with each of its values. The probability is termed a discrete probability if x can only assume the discrete values

$$x = X_1, X_2, \ldots, X_i, \ldots, X_N$$

The *discrete probability* of the event $X = x_i$ occurring is defined as $P(X_i)$.

Probability Density Functions

If x is continuous, then the *probability density function* $f(x)$ is defined so that

$$\int_{x_1}^{x_2} f(x)\,dx = \text{the probability that } x \text{ lies}$$

between x_1 and x_2. The probability is determined by defining the equation for $f(x)$ and integrating between the values of x required.

Probability Distribution Functions

The *probability distribution function* $F(X_n)$ of the discrete probability function $P(X_i)$ is defined by

$$F(X_n) = \sum_{k=1}^{n} P(X_k) = P(X_i \le X_n)$$

When x is continuous, the *probability distribution function* $F(x)$ is defined by

$$F(x_1) = \int_{-\infty}^{x_1} f(x)\,dx,$$

which implies that $F(x_1)$ is the probability that $x \le x_1$.

The *expected value* $g(x)$ of any function is defined as

$$E\{g(x)\} = \int_{-\infty}^{x} g(x)\,f(x)\,dx$$

BINOMIAL DISTRIBUTION

$F(x)$ is the probability that x will occur in n trials. If p = probability of success and q = probability of failure $= 1-p$, then

$$F(x) = C(n,x)\,p^x q^{n-x} = \frac{n!}{x!(n-x)!}\,p^x q^{n-x}$$

where

x = $0, 1, 2, \ldots, n$,
$C(n,x)$ = the number of combinations, and
n, p = parameters.

NORMAL DISTRIBUTION (Gaussian Distribution)

This is a unimodal distribution, the mode being $x = \mu$, with two points of inflection (each located at a distance σ to either side of the mode). The averages of n observations tend to become normally distributed as n increases. The variate x is said to be normally distributed if its density function $f(x)$ is given by an expression of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/2\sigma^2}, \text{ where}$$

μ = the population mean,
σ = the standard deviation of the population, and
$-\infty \le x \le \infty$.

When $\mu = 0$ and $\sigma^2 = 1 = \sigma$, the distribution is called a *unit normal* distribution. Then

$$f(x) = \frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}, \text{ where}$$

$-\infty \le x \le \infty$.

A unit normal distribution table is included in this section. In the table, the following notations are utilized:

$F(x)$ = the area under the curve from $-\infty$ to x,
$R(x)$ = the area under the curve from x to ∞, and
$W(x)$ = the area under the curve between $-x$ and x.

DISPERSION, MEAN, MEDIAN, AND MODE VALUES

If $X_1, X_2, \ldots, X_n$ represent the values of n items or observations from a population, the *arithmetic mean* of these items or observations, denoted $\bar{X}$, is defined as

$$\bar{X} = (1/n)(X_1 + X_2 + \ldots + X_n) = (1/n)\sum_{i=1}^{n} X_i$$

$\bar{X} \to \mu$ for sufficiently large values of n.

The *weighted arithmetic mean* is

$$\bar{X}_w = \frac{\Sigma\, w_i X_i}{\Sigma\, w_i}, \text{ where}$$

$\bar{X}_w$ = the weighted arithmetic mean,
X_i = the values of the observations to be averaged, and
w_i = the weight applied to the X_i value.

The *variance* of the observations is the *arithmetic mean* of the *squared deviations from the population mean*. In symbols, $X_1, X_2, \ldots, X_n$ represent the values of the n *sample* observations of a *population of size* N. If μ is the arithmetic mean of the population, the *population variance* is defined by

$$\sigma^2 = (1/N)[(X_1-\mu)^2 + (X_2-\mu)^2 + \ldots + (X_N-\mu)^2]$$
$$= (1/N)\sum_{i=1}^{N} (X_i - \mu)^2$$

The *standard deviation* of a population is

$$\sigma = \sqrt{(1/N)\,\Sigma\,(X_i-\mu)^2}$$

The *sample variance* is

$$s^2 = [1/(n-1)]\sum_{i=1}^{n}(X_i - \bar{X})^2$$

The *sample standard deviation* is

$$s = \sqrt{[1/(n-1)]\,\Sigma\,(X_i - \bar{X})^2}$$

The *coefficient of variation* $= CV = s/\bar{X}$

The *geometric mean* $= \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$

The *root-mean-squared value* $= \sqrt{(1/n)\,\Sigma\,x_i^2}$

The *median* is defined as the *value of the middle item* when the data are *rank-ordered* and the number of items is *odd*. The *median* is the *average of the middle two items* when the rank-ordered data consists of an *even* number of items.

The *mode* of a set of data is the *value that occurs with greatest frequency*.

t-DISTRIBUTION

The variate t is defined as the quotient of two independent variates x and r where x *is unit normal* and r *is the root mean square* of n other independent *unit normal variates*; that is, $t = x/r$. The following is the t-distribution with n degrees of freedom:

$$F(t) = \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)\sqrt{n\pi}}\,\frac{1}{(1+t^2/n)^{(n+1)/2}}$$

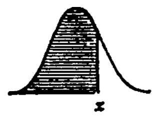
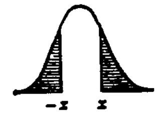
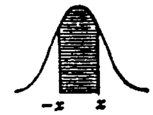
where $-\infty \le t \le \infty$.

A table is available at the end of this section which gives the values of $t_{\alpha,n}$ for values of α and n. Note that in view of the symmetry of the t-distribution, $t_{1-\alpha,n} = -t_{\alpha,n}$. The function for α follows:

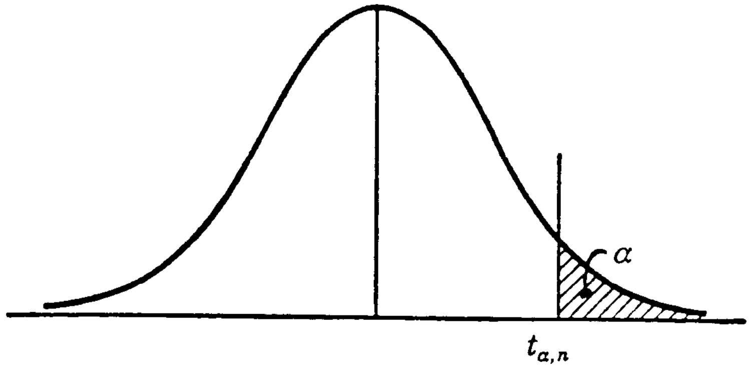
$$\alpha = \int_{t_{\alpha,n}}^{\infty} f(t)\,dt$$

A table showing "Pertinent Equations From Probability and Statistics" is included in the **INDUSTRIAL ENGINEERING SECTION** of this handbook.

UNIT NORMAL DISTRIBUTION

					
x	$f(x)$	$F(x)$	$R(x)$	$2R(x)$	$W(x)$
0.0	.3989	.5000	.5000	1.0000	0.0000
0.1	.3970	.5398	.4602	.9203	.0797
0.2	.3910	.5793	.4207	.8415	.1585
0.3	.3814	.6179	.3821	.7642	.2358
0.4	.3683	.6554	.3446	.6892	.3108
0.5	.3521	.6915	.3085	.6171	.3829
0.6	.3332	.7257	.2743	.5485	.4515
0.7	.3123	.7580	.2420	.4839	.5161
0.8	.2897	.7881	.2119	.4237	.5763
0.9	.2661	.8159	.1841	.3681	.6319
1.0	.2420	.8413	.1587	.3173	.6827
1.1	.2179	.8643	.1357	.2713	.7287
1.2	.1942	.8849	.1151	.2301	.7699
1.3	.1714	.9032	.0968	.1936	.8064
1.4	.1497	.9192	.0808	.1615	.8385
1.5	.1295	.9332	.0668	.1336	.8664
1.6	.1109	.9452	.0548	.1096	.8904
1.7	.0940	.9554	.0446	.0891	.9109
1.8	.0790	.9641	.0359	.0719	.9281
1.9	.0656	.9713	.0287	.0574	.9426
2.0	.0540	.9772	.0228	.0455	.9545
2.1	.0440	.9821	.0179	.0357	.9643
2.2	.0355	.9861	.0139	.0278	.9722
2.3	.0283	.9893	.0107	.0214	.9786
2.4	.0224	.9918	.0082	.0164	.9836
2.5	.0175	.9938	.0062	.0124	.9876
2.6	.0136	.9953	.0047	.0093	.9907
2.7	.0104	.9965	.0035	.0069	.9931
2.8	.0079	.9974	.0026	.0051	.9949
2.9	.0060	.9981	.0019	.0037	.9963
3.0	.0044	.9987	.0013	.0027	.9973
Fractiles					
1.2816	.1755	.9000	.1000	.2000	.8000
1.6449	.1031	.9500	.0500	.1000	.9000
1.9600	.0584	.9750	.0250	.0500	.9500
2.0537	.0484	.9800	.0200	.0400	.9600
2.3263	.0267	.9900	.0100	.0200	.9800
2.5758	.0145	.9950	.0050	.0100	.9900

t-DISTRIBUTION

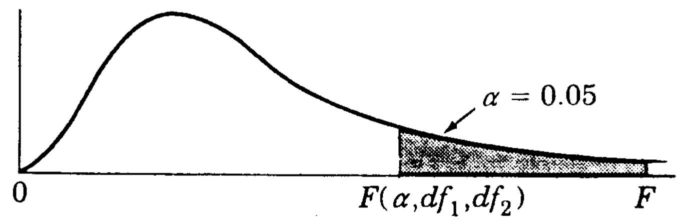


VALUES OF $t_{\alpha,n}$

n	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$	n
1	3.078	6.314	12.706	31.821	63.657	1
2	1.886	2.920	4.303	6.965	9.925	2
3	1.638	2.353	3.182	4.541	5.841	3
4	1.533	2.132	2.776	3.747	4.604	4
5	1.476	2.015	2.571	3.365	4.032	5
6	1.440	1.943	2.447	3.143	3.707	6
7	1.415	1.895	2.365	2.998	3.499	7
8	1.397	1.860	2.306	2.896	3.355	8
9	1.383	1.833	2.262	2.821	3.250	9
10	1.372	1.812	2.228	2.764	3.169	10
11	1.363	1.796	2.201	2.718	3.106	11
12	1.356	1.782	2.179	2.681	3.055	12
13	1.350	1.771	2.160	2.650	3.012	13
14	1.345	1.761	2.145	2.624	2.977	14
15	1.341	1.753	2.131	2.602	2.947	15
16	1.337	1.746	2.120	2.583	2.921	16
17	1.333	1.740	2.110	2.567	2.898	17
18	1.330	1.734	2.101	2.552	2.878	18
19	1.328	1.729	2.093	2.539	2.861	19
20	1.325	1.725	2.086	2.528	2.845	20
21	1.323	1.721	2.080	2.518	2.831	21
22	1.321	1.717	2.074	2.508	2.819	22
23	1.319	1.714	2.069	2.500	2.807	23
24	1.318	1.711	2.064	2.492	2.797	24
25	1.316	1.708	2.060	2.485	2.787	25
26	1.315	1.706	2.056	2.479	2.779	26
27	1.314	1.703	2.052	2.473	2.771	27
28	1.313	1.701	2.048	2.467	2.763	28
29	1.311	1.699	2.045	2.462	2.756	29
inf.	1.282	1.645	1.960	2.326	2.576	inf.

Critical Values of F

For a particular combination of numerator and denominator degrees of freedom, entry represents the critical values of F corresponding to a specified upper tail area (α).



Denominator df_2	Numerator df_1																		
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**12**	**15**	**20**	**24**	**30**	**40**	**60**	**120**	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	**6.61**	**5.79**	**5.41**	**5.19**	**5.05**	**4.95**	**4.88**	**4.82**	**4.77**	**4.74**	**4.68**	**4.62**	**4.56**	**4.53**	**4.50**	**4.46**	**4.43**	**4.40**	**4.36**
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	**4.96**	**4.10**	**3.71**	**3.48**	**3.33**	**3.22**	**3.14**	**3.07**	**3.02**	**2.98**	**2.91**	**2.85**	**2.77**	**2.74**	**2.70**	**2.66**	**2.62**	**2.58**	**2.54**
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	**4.54**	**3.68**	**3.29**	**3.06**	**2.90**	**2.79**	**2.71**	**2.64**	**2.59**	**2.54**	**2.48**	**2.40**	**2.33**	**2.29**	**2.25**	**2.20**	**2.16**	**2.11**	**2.07**
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	**4.35**	**3.49**	**3.10**	**2.87**	**2.71**	**2.60**	**2.51**	**2.45**	**2.39**	**2.35**	**2.28**	**2.20**	**2.12**	**2.08**	**2.04**	**1.99**	**1.95**	**1.90**	**1.84**
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	**4.24**	**3.39**	**2.99**	**2.76**	**2.60**	**2.49**	**2.40**	**2.34**	**2.28**	**2.24**	**2.16**	**2.09**	**2.01**	**1.96**	**1.92**	**1.87**	**1.82**	**1.77**	**1.71**
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	**4.17**	**3.32**	**2.92**	**2.69**	**2.53**	**2.42**	**2.33**	**2.27**	**2.21**	**2.16**	**2.09**	**2.01**	**1.93**	**1.89**	**1.84**	**1.79**	**1.74**	**1.68**	**1.62**
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

DIFFERENTIAL CALCULUS

The Derivative

For any function $y = f(x)$,

the derivative $= D_x y = dy/dx = y'$

$$y' = \lim_{\Delta x \to 0} [(\Delta y)/(\Delta x)]$$
$$= \lim_{\Delta x \to 0} \{[f(x + \Delta x) - f(x)]/(\Delta x)\}$$

y' = the slope of the curve $f(x)$.

TEST FOR A MAXIMUM

$y = f(x)$ is a maximum for

$x = a$, if $f'(a) = 0$ and $f''(a) < 0$.

TEST FOR A MINIMUM

$y = f(x)$ is a minimum for

$x = a$, if $f'(a) = 0$ and $f''(a) > 0$.

TEST FOR A POINT OF INFLECTION

$y = f(x)$ has a point of inflection at $x = a$,

if $f''(a) = 0$, and

if $f''(x)$ changes sign as x increases through $x = a$.

The Partial Derivative

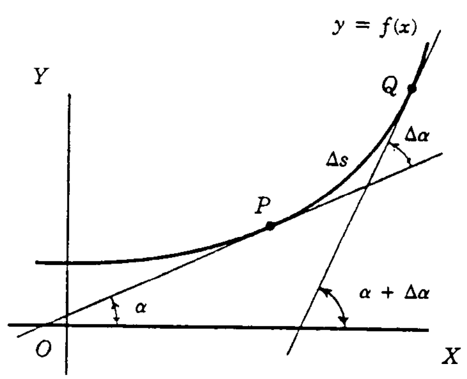
In a function of two independent variables x and y, a derivative with respect to one of the variables may be found if the other variable is *assumed* to remain constant. If *y is kept fixed*, the function

$$z = f(x, y)$$

becomes a function of the *single variable* x, and its derivative (if it exists) can be found. This derivative is called the *partial derivative of z with respect to x*. The partial derivative with respect to x is denoted as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x}$$

The Curvature of Any Curve ♦



The curvature K of a curve at P is the limit of its average curvature for the arc PQ as Q approaches P. This is also expressed as: the curvature of a curve at a given point is the rate-of-change of its inclination with respect to its arc length.

$$K = \lim_{\Delta s \to 0} \frac{\Delta \alpha}{\Delta s} = \frac{d\alpha}{ds}$$

CURVATURE IN RECTANGULAR COORDINATES

$$K = \frac{y''}{[1 + (y')^2]^{3/2}}$$

When it may be easier to differentiate the function with respect to y rather than x, the notation x' will be used for the derivative.

$$x' = dx/dy$$

$$K = \frac{-x''}{[1 + (x')^2]^{3/2}}$$

THE RADIUS OF CURVATURE

The *radius of curvature* R at any point on a curve is defined as the absolute value of the reciprocal of the curvature K at that point.

$$R = \frac{1}{|K|} \qquad (K \neq 0)$$

$$R = \frac{[1 + (y')^2]^{3/2}}{|y''|} \qquad (y'' \neq 0)$$

L'Hospital's Rule (L'Hôpital's Rule)

If the fractional function $f(x)/g(x)$ assumes one of the indeterminate forms $0/0$ or ∞/∞ (where a is finite or infinite), then

$$\lim_{x \to a} f(x)/g(x)$$

is equal to the first of the expressions

$$\lim_{x \to a} \frac{f'(x)}{g'(x)},\ \lim_{x \to a} \frac{f''(x)}{g''(x)},\ \lim_{x \to a} \frac{f'''(x)}{g'''(x)}$$

which is not indeterminate, provided such first indicated limit exists.

INTEGRAL CALCULUS

Fundamental Theorem

The *fundamental theorem* of the integral calculus is:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x_i = \int_a^b f(x)\, dx$$

Also, $\Delta x_i \to 0$ for all i.

A table of derivatives and integrals is available on the next page. The integral equations can be used along with the following methods of integration:

A. Integration by Parts (integral equation #6),

B. Integration by Substitution, and

C. Separation of Rational Fractions into Partial Fractions.

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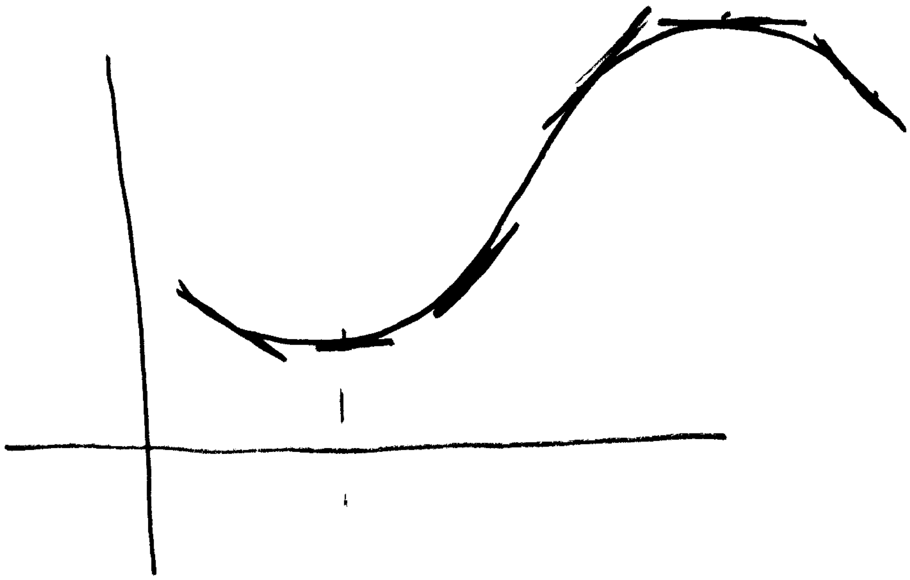
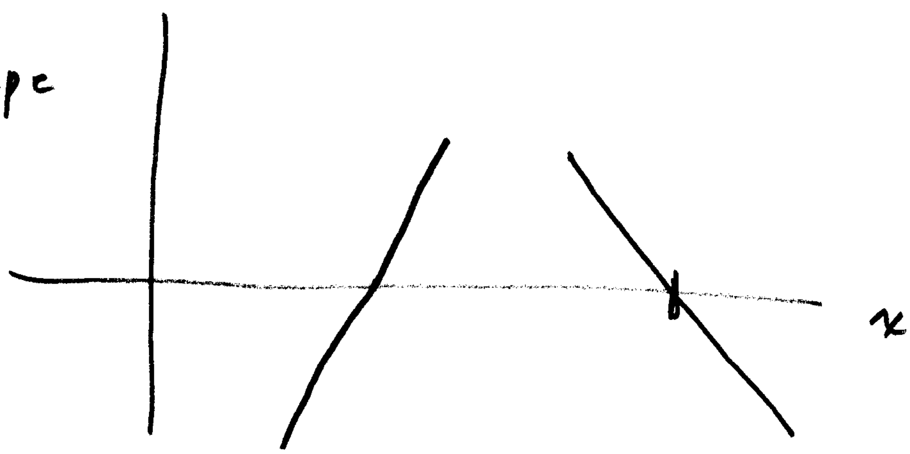


slope



a

$a+d$

$a+2d$

$\vdots$

a

ar

ar^2

$\vdots$

ar^{n-1}

DERIVATIVES and INDEFINITE INTEGRALS

In these formulas, u, v, and w represent functions of x. Also, a, c, and n represent constants. All arguments of the trigonometric functions are in radians. A constant of integration should be added to the integrals. To avoid terminology difficulty, the following definitions are followed: $\arcsin u = \sin^{-1} u$, $(\sin u)^{-1} = 1/\sin u$.

1. $dc/dx = 0$
2. $dx/dx = 1$
3. $d(cu)/dx = c\, du/dx$
4. $d(u + v - w)/dx = du/dx + dv/dx - dw/dx$
5. $d(uv)/dx = u\, dv/dx + v\, du/dx$
6. $d(uvw)/dx = uv\, dw/dx + uw\, dv/dx + vw\, du/dx$
7. $\dfrac{d(u/v)}{dx} = \dfrac{v\, du/dx - u\, dv/dx}{v^2}$
8. $d(u^n)/dx = nu^{n-1}\, du/dx$
9. $d[f(u)]/dx = \{d[f(u)]/du\}\, du/dx$
10. $du/dx = 1/(dx/du)$
11. $\dfrac{d(\log_a u)}{dx} = (\log_a e)\, \dfrac{1}{u}\dfrac{du}{dx}$
12. $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$
13. $\dfrac{d(a^u)}{dx} = (\ln a)\, a^u \dfrac{du}{dx}$
14. $d(e^u)/dx = e^u\, du/dx$
15. $d(u^v)/dx = vu^{v-1}\, du/dx + (\ln u)\, u^v\, dv/dx$
16. $d(\sin u)/dx = \cos u\, du/dx$
17. $d(\cos u)/dx = -\sin u\, du/dx$
18. $d(\tan u)/dx = \sec^2 u\, du/dx$
19. $d(\cot u)/dx = -\csc^2 u\, du/dx$
20. $d(\sec u)/dx = \sec u \tan u\, du/dx$
21. $d(\csc u)/dx = -\csc u \cot u\, du/dx$
22. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ $\quad (-\pi/2 \le \sin^{-1} u \le \pi/2)$
23. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ $\quad (0 \le \cos^{-1} u \le \pi)$
24. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$ $\quad (-\pi/2 < \tan^{-1} u < \pi/2)$
25. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$ $\quad (0 < \cot^{-1} u < \pi)$
26. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$ $\quad (0 \le \sec^{-1} u < \pi/2)(-\pi \le \sec^{-1} u < -\pi/2)$
27. $\dfrac{d(\csc^{-1} u)}{dx} = -\dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$ $\quad (0 < \csc^{-1} u \le \pi/2)(-\pi < \csc^{-1} u \le -\pi/2)$

1. $\int df(x) = f(x)$
2. $\int dx = x$
3. $\int a f(x)\, dx = a \int f(x)\, dx$
4. $\int [u(x) \pm v(x)]\, dx = \int u(x)\, dx \pm \int v(x)\, dx$
5. $\int x^m\, dx = \dfrac{x^{m+1}}{m+1}$ $\quad (m \ne -1)$
6. $\int u(x)\, dv(x) = u(x)v(x) - \int v(x)\, du(x)$
7. $\int \dfrac{dx}{ax+b} = \dfrac{1}{a} \ln |ax+b|$
8. $\int \dfrac{dx}{\sqrt{x}} = 2\sqrt{x}$
9. $\int a^x\, dx = \dfrac{a^x}{\ln a}$
10. $\int \sin x\, dx = -\cos x$
11. $\int \cos x\, dx = \sin x$
12. $\int \sin^2 x\, dx = \dfrac{x}{2} - \dfrac{\sin 2x}{4}$
13. $\int \cos^2 x\, dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4}$
14. $\int x \sin x\, dx = \sin x - x \cos x$
15. $\int x \cos x\, dx = \cos x + x \sin x$
16. $\int \sin x \cos x\, dx = (\sin^2 x)/2$
17. $\int \sin ax \cos bx\, dx = -\dfrac{\cos (a-b)x}{2(a-b)} - \dfrac{\cos (a+b)x}{2(a+b)}$ $\quad (a^2 \ne b^2)$
18. $\int \tan x\, dx = -\ln |\cos x| = \ln |\sec x|$
19. $\int \cot x\, dx = -\ln |\csc x| = \ln |\sin x|$
20. $\int \tan^2 x\, dx = \tan x - x$
21. $\int \cot^2 x\, dx = -\cot x - x$
22. $\int e^{ax}\, dx = (1/a)\, e^{ax}$
23. $\int x e^{ax}\, dx = (e^{ax}/a^2)(ax-1)$
24. $\int \ln x\, dx = x[\ln(x) - 1]$ $\quad (x > 0)$
25. $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$ $\quad (a \ne 0)$
26. $\int \dfrac{dx}{ax^2+c} = \dfrac{1}{\sqrt{ac}}\tan^{-1}\left(x\sqrt{\dfrac{a}{c}}\right)$, $\quad (a > 0, c > 0)$

27a. $\int \dfrac{dx}{ax^2+bx+c} = \dfrac{2}{\sqrt{4ac-b^2}}\tan^{-1}\dfrac{2ax+b}{\sqrt{4ac-b^2}}$ $\quad (4ac - b^2 > 0)$

27b. $\int \dfrac{dx}{ax^2+bx+c} = \dfrac{1}{\sqrt{b^2-4ac}} \ln \left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|$ $\quad (b^2 - 4ac > 0)$

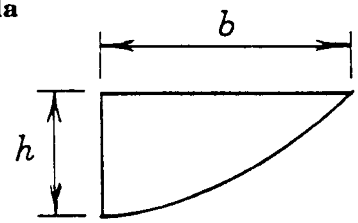
27c. $\int \dfrac{dx}{ax^2+bx+c} = -\dfrac{2}{2ax+b}$, $\quad (b^2 - 4ac = 0)$

MENSURATION OF AREAS AND VOLUMES

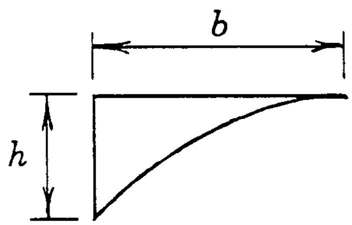
Nomenclature

A = total surface area
p = perimeter
V = volume

Parabola



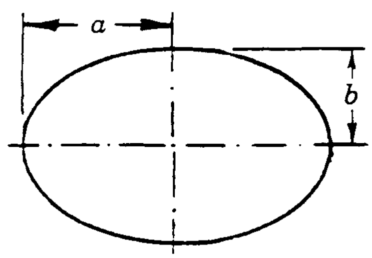
$$A = 2bh/3$$



$$A = bh/3$$

Ellipse

♦



$$A = \pi ab$$

$$p_{\text{approx}} = 2\pi\sqrt{(a^2 + b^2)/2}$$

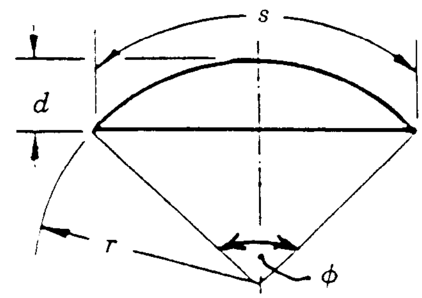
$$p = \pi(a + b)[1 + (^1/_2)^2\lambda^2 + (^1/_2 \times {}^1/_4)^2\lambda^4 + (^1/_2 \times {}^1/_4 \times {}^3/_6)^2\lambda^6 + (^1/_2 \times {}^1/_4 \times {}^3/_6 \times {}^5/_8)^2\lambda^8 + (^1/_2 \times {}^1/_4 \times {}^3/_6 \times {}^5/_8 \times {}^7/_{10})^2\lambda^{10} + \ldots],$$

where

$$\lambda = (a - b)/(a + b)$$

Circular Segment

♦

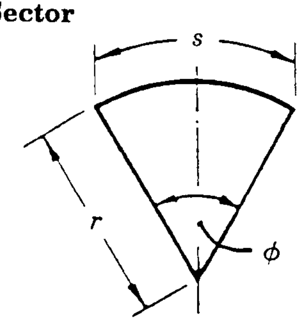


$$A = [r^2(\phi - \sin \phi)]/2$$

$$\phi = s/r = 2\{\arccos[(r - d)/r]\}$$

Circular Sector

♦

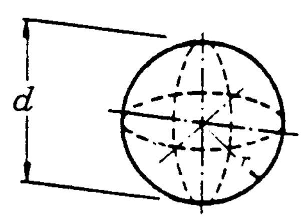


$$A = \phi r^2/2 = sr/2$$

$$\phi = s/r$$

Sphere

♦



$$V = 4\pi r^3/3 = \pi d^3/6$$

$$A = 4\pi r^2 = \pi d^2$$

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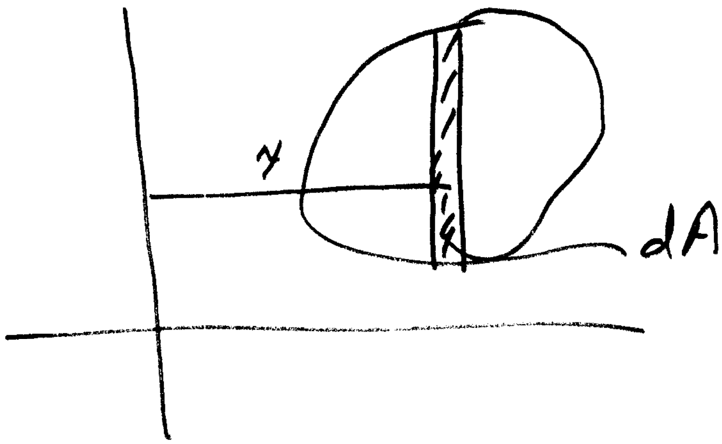
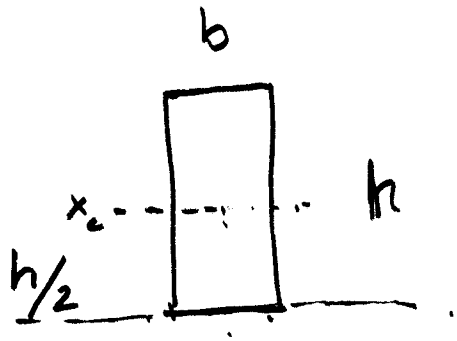


$$A = \int dA$$

$$M_y = \int dM_y = \int x\, dA$$

$$x_c = \frac{M_y}{A} = \frac{\int x\, dA}{A}$$



$$I = \frac{bh^3}{12}$$

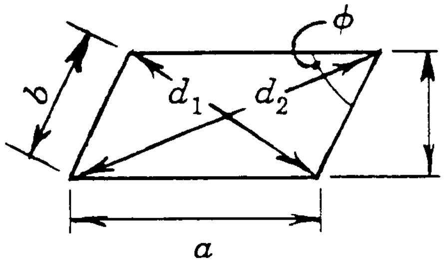
$$I_{x_c} = \frac{bh^3}{12}$$

$$I_x = \frac{bh^3}{12} + (bh)\left(\frac{h}{2}\right)^2 = \frac{bh^3}{12} + \frac{bh^3}{4}$$

$$= \frac{bh^3}{3}$$

MENSURATION OF AREAS AND VOLUMES

Parallelogram

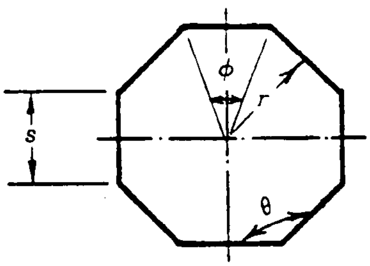


$$p = 2(a + b)$$
$$d_1 = \sqrt{a^2 + b^2 - 2ab(\cos \phi)}$$
$$d_2 = \sqrt{a^2 + b^2 + 2ab(\cos \phi)}$$
$$d_1^2 + d_2^2 = 2(a^2 + b^2)$$
$$A = ah = ab(\sin \phi)$$

If $a = b$, the parallelogram is a rhombus.

Regular Polygon (n equal sides)

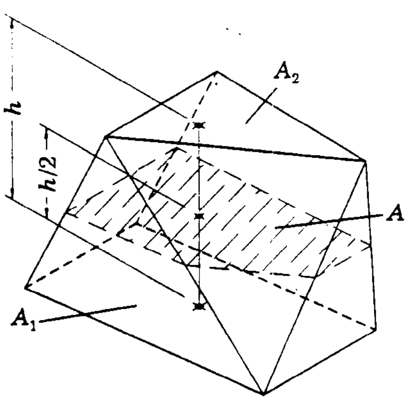
♦



$$\phi = 2\pi/n$$
$$\theta = [\pi(n-2)]/n = \pi - \phi$$
$$p = ns$$
$$s = 2r[\tan(\phi/2)]$$
$$A = (nsr)/2$$

Prismoid

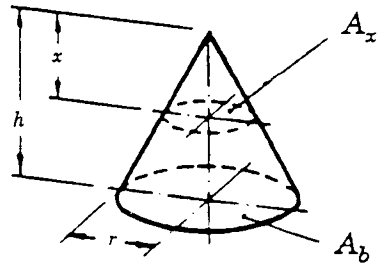
♦



$$V = (h/6)(A_1 + A_2 + 4A)$$

Right Circular Cone

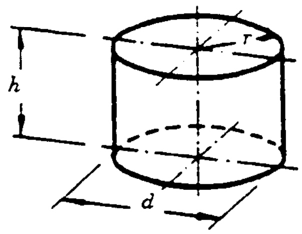
♦



$$V = (\pi r^2 h)/3$$
$$A = \text{side area} + \text{base area}$$
$$= \pi r(r + \sqrt{r^2 + h^2})$$
$$A_x : A_b = x^2 : h^2$$

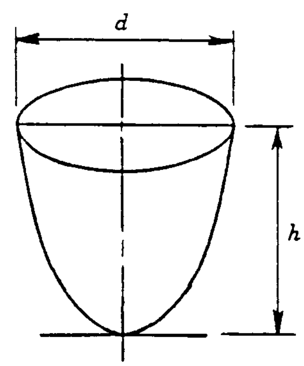
Right Circular Cylinder

♦



$$V = \pi r^2 h = \pi d^2 h/4$$
$$A = \text{side area} + \text{end areas}$$
$$= 2\pi r(h + r)$$

Paraboloid of Revolution



$$V = \pi d^2 h/8$$

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CENTROIDS AND MOMENTS OF INERTIA

The *location of the centroid of an area*, bounded by the axes and the function $y = f(x)$, can be found by integration.

$$x_c = \frac{\int x\,dA}{A}$$

$$y_c = \frac{\int y\,dA}{A}$$

$$A = \int f(x)\,dx$$

$$dA = f(x)\,dx = g(y)\,dy$$

The *first moment of area* with respect to the y-axis and the x-axis, respectively, are:

$$M_y = \int x\,dA = x_c A$$

$$M_x = \int y\,dA = y_c A$$

The *moment of inertia* (*second moment of area*) with respect to the y-axis and the x-axis, respectively, are:

$$I_y = \int x^2\,dA$$

$$I_x = \int y^2\,dA$$

The moment of inertia taken with respect to an axis passing through the area's centroid is the *centroidal moment of inertia*. The *parallel axis theorem* for the moment of inertia with respect to another axis parallel with and located d units from the centroidal axis is expressed by

$$I_{\text{parallel axis}} = I_c + Ad^2$$

Values for standard shapes are presented in a table in the **DYNAMICS** section.

DIFFERENTIAL EQUATIONS

A common class of ordinary linear differential equations is

$$b_N \frac{d^N y(x)}{dx^N} + \dots + b_1 \frac{dy(x)}{dx} + b_0 y(x) = f(x)$$

where $b_N, \dots, b_n, \dots, b_1, b_0$ are constants.

When the equation is a homogeneous differential equation, $f(x) = 0$, the solution is

$$y_h(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \dots + C_n e^{r_n x} + \dots + C_N e^{r_N x}$$

where r_n is the nth distinct root of the characteristic polynomial $P(x)$ with

$$P(r) = b_N r^N + b_{N-1} r^{N-1} + \dots + b_1 r + b_0$$

If the root $r_1 = r_2$, then $C_2 e^{r_2 x}$ is replaced with $C_2 x e^{r_1 x}$. Higher orders of multiplicity imply higher powers of x. The complete solution for the differential equation is

$$y(x) = y_h(x) + y_p(x),$$

where $y_p(x)$ is any solution with $f(x)$ present. If $f(x)$ has $e^{r_n x}$ terms, then resonance is manifested. Furthermore, specific $f(x)$ forms result in specific $y_p(x)$ forms, some of which are:

$f(x)$	$y_p(x)$
A	B
$Ae^{\alpha x}$	$Be^{\alpha x}$, $\alpha \neq r_n$
$A_1 \sin \omega x + A_2 \cos \omega x$	$B_1 \sin \omega x + B_2 \cos \omega x$

If the independent variable is time t, then transient dynamic solutions are implied.

First Order Linear Homogeneous Differential Equations With Constant Coefficients

$y' + ay = 0$, where a is a real constant:

Solution, $y = Ce^{-at}$, where

C = a constant that satisfies the initial conditions.

Second Order Linear Homogeneous Differential Equations With Constant Coefficients

An equation of the form

$$y'' + 2ay' + by = 0$$

can be solved by the method of undetermined coefficients where a solution of the form $y = Ce^{rx}$ is sought. Substitution of this solution gives

$$(r^2 + 2ar + b)Ce^{rx} = 0$$

and since Ce^{rx} cannot be zero, the characteristic equation must vanish or

$$r^2 + 2ar + b = 0$$

The roots of the characteristic equation are

$$r_{1,2} = -a \pm \sqrt{a^2 - b}$$

and can be real and distinct for $a^2 > b$, real and equal for $a^2 = b$, and complex for $a^2 < b$.

If $a^2 > b$, the solution is of the form (overdamped)

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

If $a^2 = b$, the solution is of the form (critically damped)

$$y = (C_1 + C_2 x)e^{r_1 x}$$

If $a^2 < b$, the solution is of the form (underdamped)

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

where

$$\alpha = -a$$

$$\beta = \sqrt{b - a^2}$$

FOURIER SERIES

Every function $F(t)$, which has the period $\tau = 2\pi/\omega$ and satisfies certain continuity conditions, can be represented by a series plus a constant.

$$F(t) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

The above equation holds if $F(t)$ has a continuous derivative $F'(t)$ for all t. Multiply both sides of the equation by $\cos m\omega t$ and integrate from 0 to τ.

$$K_2 \frac{d^2y}{dx^2} + K_1 \frac{dy}{dx} + K_0 y = f(x)$$

$$y'' + 12y' + 100y = 0$$

$$y = Ce^{rx}$$

$$r^2 + 12r + 100 = 0$$

$$r = \frac{-12 \mp \sqrt{144 - 400}}{2}$$

$$= \frac{-12 \mp \sqrt{-256}}{2} = \frac{-12 \mp 16i}{2}$$

$$r = -6 \mp 8i$$

$$y = C_1 e^{(-6+8i)x} + C_2 e^{(-6-8i)x} = e^{-6x}\left[C_1 e^{8ix} + C_2 e^{-8ix}\right]$$

$$y = e^{-6x}(C_1 \cos 8x + C_2 \sin 8x)$$

$$y'' + 3y' + 2y = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+2)(r+1) = 0 \quad , \quad r = -1, -2$$

$$\underline{y = C_1 e^{-1x} + C_2 e^{-2x}}$$

$$y'' + 6y' + 9y = 0$$

$$r^2 + 6r + 9 = 0$$

$$(r+3)(r+3) = 0 \quad , \quad r = -3, -3$$

$$\underline{y = C_1 e^{-3x} + C_2 x e^{-3x}}$$

$$y'' + 100y = 0$$

$$r^2 + 100 = 0 \quad , \quad r^2 = -100$$

$$r = \mp 10i$$

$$y = C_1 e^{10ix} + C_2 e^{-10ix}$$

$$y = C_1 \sin 10x + C_2 \cos 10x$$

$$\int_0^{\tau} F(t)\cos m\omega t\,dt = \int_0^{\tau}(a_o/2)\cos m\omega t\,dt + \sum_{n=1}^{\infty}\left[a_n\int_0^{\tau}\cos n\omega t\cos m\omega t\,dt + b_n\int_0^{\tau}\sin n\omega t\cos m\omega t\,dt\right]$$

Term-by-term integration of the series can be justified if $F(t)$ is continuous. The *coefficients* are

$$a_n = (2/\tau)\int_0^{\tau} F(t)\cos n\omega t\,dt \quad \text{and}$$

$$b_n = (2/\tau)\int_0^{\tau} F(t)\sin n\omega t\,dt, \quad \text{where}$$

$\tau = 2\pi/\omega$. The constants a_n, b_n are the *Fourier coefficients* of $F(t)$ for the interval 0 to τ, and the corresponding series is called the *Fourier series* of $F(t)$ over the same interval. The integrals have the same value over any interval of length τ.

FOURIER TRANSFORM

The Fourier transform pair, one form of which is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-j\omega t}\,dt$$

$$f(t) = [1/(2\pi)]\int_{-\infty}^{\infty} F(\omega)\,e^{j\omega t}\,d\omega$$

can be used to characterize a broad class of signal models in terms of their frequency or spectral content. Some useful transform pairs are:

$f(t)$	$F(\omega)$
$\delta(t)$	1
$u(t)$	$(1/2)\delta(\omega) + 1/j\omega$
$u\left(t+\frac{\tau}{2}\right)-u\left(t-\frac{\tau}{2}\right) = r_{\text{rect}}\left(\frac{t}{\tau}\right)$	$\tau\,\dfrac{\sin(\omega\tau/2)}{\omega\tau/2}$
$e^{j\omega_o t}$	$2\pi\delta(\omega-\omega_o)$

Some mathematical liberties are required to obtain the second and fourth form. Other Fourier transforms are derivable from the Laplace transform by replacing s with $j\omega$ provided

$$f(t) = 0, \quad t < 0$$

$$\int_0^{\infty} |f(t)|\,dt < \infty$$

LAPLACE TRANSFORMS

The unilateral Laplace transform pair

$$F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$$

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}\,ds$$

represents a powerful tool for the transient and frequency response of linear time invariant systems. Some useful Laplace transform pairs are [Note: The last two transforms represent the Final Value Theorem (F.V.T.) and Initial Value Theorem (I.V.T.) respectively. It is assumed that the limits exist.]:

$f(t)$	$F(s)$
$\delta(t)$, Impulse at $t = 0$	1
$u(t)$, Step at $t = 0$	$1/s$
$t[u(t)]$, Ramp at $t = 0$	$1/s^2$
$e^{-\alpha t}$	$1/(s+\alpha)$
$t\,e^{-\alpha t}$	$1/(s+\alpha)^2$
$e^{-\alpha t}\sin\beta t$	$\beta/[(s+\alpha)^2+\beta^2]$
$e^{-\alpha t}\cos\beta t$	$(s+\alpha)/[(s+\alpha)^2+\beta^2]$
$\mathcal{L}\left\{\dfrac{d^n f(t)}{dt^n}\right\}$	$s^n F(s) - \sum_{m=0}^{n-1} s^{n-m-1}\dfrac{d^m f(0)}{dt^m}$
$\int_0^t f(\tau)\,d\tau$	$(1/s)F(s)$
$\int_0^t x(t-\tau)\,h(t)\,d\tau$	$H(s)X(s)$
$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$
$\lim_{t\to 0} f(t)$	$\lim_{s\to\infty} sF(s)$

DIFFERENCE EQUATIONS

Difference equations are used to model discrete systems. Systems which can be described by difference equations include: computer program variables iteratively evaluated in a loop, sequential circuits, cash flows, recursive processes, systems with time-delay components, etc. Any system whose input $v(t)$ and output $y(t)$ are defined only at the equally-spaced intervals $t = kT$ can be described by a difference equation.

First Order Linear Difference Equation

The difference equation

$$P_k = P_{k-1}(1+i) - A$$

represents the balance P of a loan after the kth payment A. If P_k is defined as $y(k)$, the model becomes

$$y(k) - (1+i)y(k-1) = -A$$

Second Order Linear Difference Equation

The Fibonacci number sequence can be generated by

$$y(k) = y(k-1) + y(k-2)$$

where $y(-1) = 1$ and $y(-2) = 1$. An alternate form for this model is

$$f(x+2) = f(k+1) + f(k)$$

with $f(0) = 1$ and $f(1) = 1$.

z - Transforms

The z-transform pair

$$F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}$$

$$f(k) = \frac{1}{2\pi i}\oint_{\Gamma} F(z)z^{n-1}\,dz$$

represents a powerful tool for solving linear shift invariant difference equations. A limited unilateral list of z-transform pairs follows [Note: The last two transform pairs represent the Initial Value Theorem (I.V.T.) and the Final Value Theorem (F.V.T.) respectively.]:

$f(k)$	$F(z)$
$\delta(k)$, Impulse at $k = 0$	1
$u(k)$, Step at $k = 0$	$1/(1 - z^{-1})$
β^k	$1/(1 - \beta z^{-1})$
$y(k-1)$	$z^{-1}Y(z) + y(-1)$
$y(k-2)$	$z^{-2}Y(z) + y(-2) + y(-1)z^{-1}$
$y(k+1)$	$zY(z) - zy(0)$
$y(k+2)$	$z^2Y(z) - z^2y(0) - zy(1)$
$\sum_{m=0}^{\infty} X(k-m)h(m)$	$H(z)X(z)$
$\lim_{k\to 0} f(k)$	$\lim_{z\to\infty} F(z)$
$\lim_{k\to\infty} f(k)$	$\lim_{z\to 1} (1 - z^{-1})F(z)$

EULER'S APPROXIMATION

$$x_{i+1} = x_i + \Delta t(dx_i/dt)$$

NUMERICAL METHODS

Newton's Method of Root Extraction

Given a polynomial $P(x)$ with n simple roots, $a_1, a_2, \cdots, a_n$ where

$$P(x) = \prod_{m=1}^{n} (x - a_m) = x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_{n-1}$$

and $P(a_i) = 0$. A root a_i can be computed by the iterative algorithm

$$a_i^{j+1} = a_i^j - \left.\frac{P(x)}{\partial P(x)/\partial x}\right|_{x = a_i^j}$$

with $|P(a_i^{j+1})| \le |P(a_i^j)|$. Convergence is quadratic.

Newton's Method of Minimization

Given a scalar value function

$$h(\boldsymbol{x}) = h(x_1, x_2, \cdots, x_n)$$

find a vector $\boldsymbol{x}^* \epsilon R_n$ such that

$$h(\boldsymbol{x}^*) \le h(\boldsymbol{x}) \text{ for all } \boldsymbol{x}$$

Newton's algorithm is

$$\boldsymbol{x}_{K+1} = \boldsymbol{x}_K - \left(\left.\frac{\partial^2 h}{\partial \boldsymbol{x}^2}\right|_{\boldsymbol{x} = \boldsymbol{x}_K}\right)^{-1} \left.\frac{\partial h}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x} = \boldsymbol{x}_K}$$

where

$$\frac{\partial h}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial h}{\partial x_1} \\ \frac{\partial h}{\partial x_2} \\ \cdots \\ \cdots \\ \frac{\partial h}{\partial x_n} \end{bmatrix}$$

and

$$\frac{\partial^2 h}{\partial \boldsymbol{x}^2} = \begin{bmatrix} \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 h}{\partial x_1 \partial x_2} & \cdots & \cdots & \cdots & \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 h}{\partial x_2^2} & \cdots & \cdots & \cdots & \frac{\partial^2 h}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 h}{\partial x_1 \partial x_n} & \frac{\partial^2 h}{\partial x_2 \partial x_n} & \cdots & \cdots & \cdots & \frac{\partial^2 h}{\partial x_n^2} \end{bmatrix}$$

Numerical Integration

Three of the more common numerical integration algorithms used to evaluate the integral

$$\int_a^b f(x)\,dx$$

with $\Delta x = (b - a)/n$ are:

Euler's or Forward Rectangular Rule

$$\int_a^b f(x)\,dx \approx \Delta x \sum_{k=0}^{n-1} f(a + k\Delta x)$$

Trapezoidal Rule

for $n = 1$

$$\int_a^b f(x)\,dx \approx \Delta x \left[\frac{f(a) + f(b)}{2}\right]$$

for $n > 1$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(a) + 2\sum_{k=1}^{n-1} f(a + k\Delta x) + f(b)\right]$$

Simpson's Rule/Parabolic Rule (n must be an even integer)

for $n = 2$

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

for $n \ge 4$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(a) + 2\sum_{k=2,4,6,\cdots}^{n-2} f(a + k\Delta x) + 4\sum_{k=1,3,5,\cdots}^{n-1} f(a + k\Delta x) + f(b)\right]$$

Numerical Solution of Ordinary Differential Equations

Given a differential equation

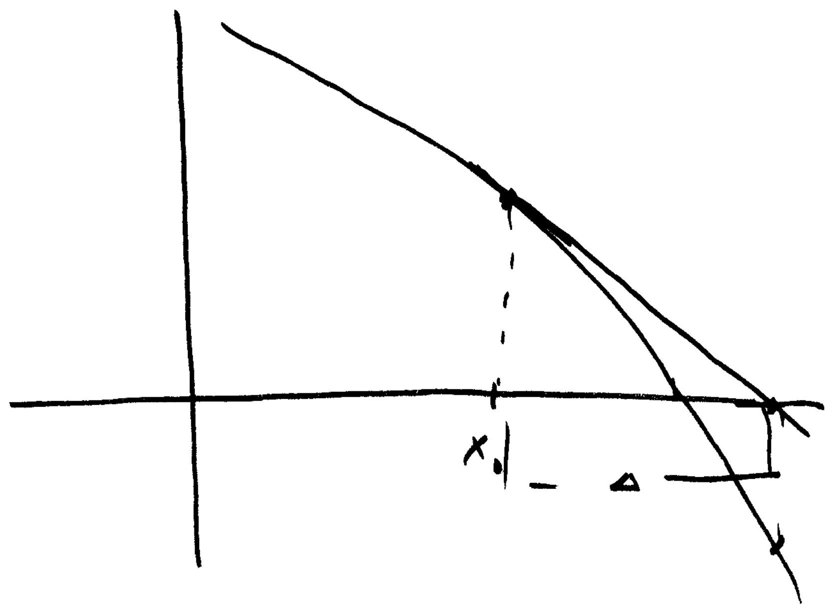
$$dy/dt = f(y,t) \quad \text{with} \quad y(0) = y_o$$

At some general time $k\Delta t$

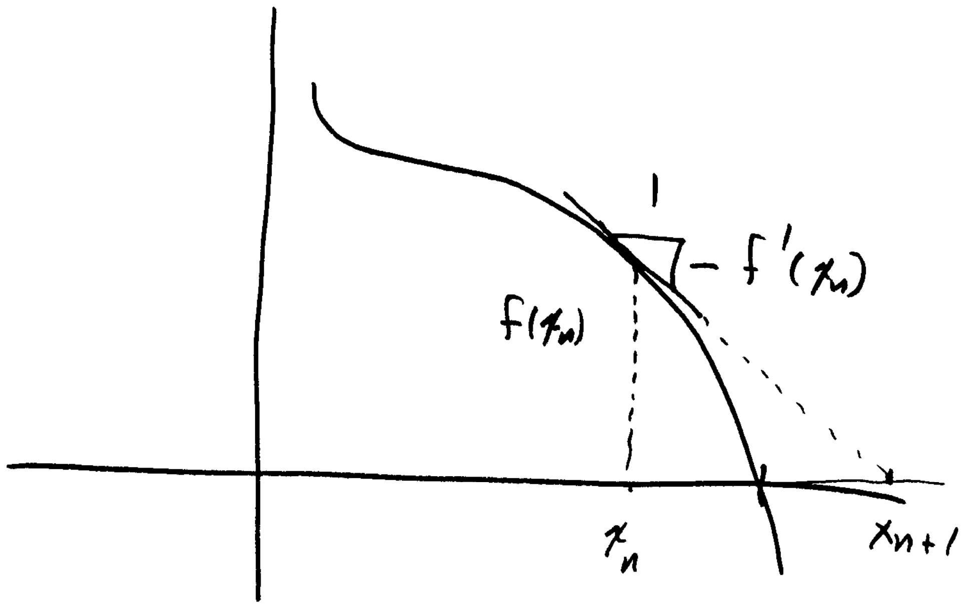
$$y[(k+1)\Delta t] \cong y(k\Delta t) + \Delta t\, f[y(k\Delta t), k\Delta t]$$

which can be used with starting condition y_o to solve recursively for $y(\Delta t), y(2\Delta t), \cdots, y(n\Delta t)$.

The method can be extended to nth order differential equations by recasting them as n first order equations.



$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

NATIONAL COUNCIL OF EXAMINERS FOR ENGINEERING AND SURVEYING

FUNDAMENTALS OF ENGINEERING

(FE)

SAMPLE QUESTIONS

First Edition - March 1996

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60 Questions

Directions: Each of the questions or incomplete statements below is followed by four suggested answers or completions. Select the one that is best in each case and then blacken the corresponding space on the answer sheet.

1. The partial derivative $\frac{\partial y}{\partial x}$ of $y = x^2z + 3z^2x + 6(x+z)$ is:

(A) $2xz + 3z^2 + 6$
(B) $x^2z + 6zx + 6z$
(C) $2x + 9$
(D) $2x + 6z + 6$

2. A bag contains 100 balls numbered from 1 to 100. One ball is removed. What is the probability that the number on this ball is odd or greater than 80?

(A) 0.2
(B) 0.5
(C) 0.6
(D) 0.8

3. Consider a function of x equal to the determinant shown below.

$$f(x) = \begin{vmatrix} x & x^2 \\ x^4 & x^3 \end{vmatrix}$$

The first derivative $f'(x)$ of this function with respect to x is equal to:

(A) $3x^2 - 8x^4$
(B) $4x^3 - 6x^5$
(C) $x^4 - x^6$
(D) $3x^4 - 5x^6$

4. The Laplace transform of a step function of magnitude a is:

(A) $\frac{1}{s+a}$

(B) $\frac{a}{s}$

(C) $\frac{a}{s+a}$

(D) $\frac{a}{s^2}$

GO ON TO THE NEXT PAGE.

Mathematics

1. Given $y = x^2 z + 3z^2 x + 6(x+z)$

Determine $\partial y / \partial x$

$$\frac{\partial y}{\partial x} = 2xz + 3z^2 + 6$$ Answer A

2. Balls numbered 1 to 100. One drawn. Determine probability that the number is odd or greater than 80.

$P(odd) = 50/100 = 1/2$, $P(>80) = \frac{20}{100}$

$P(odd\ and >80) = 10/100$

$P(odd\ or >80) = P(odd) + P(>80) - P(odd\ \&\ >80)$

$$= \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{60}{100}$$

$P(odd\ or >80) = 0.6$ Answer C

(Property 2 - page 7)

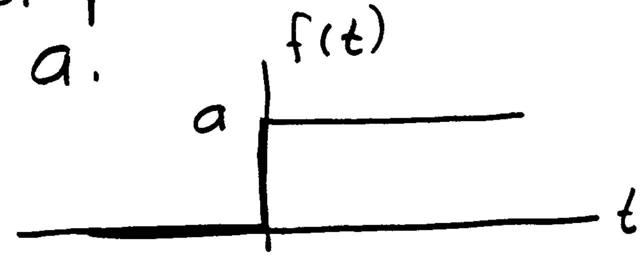
JW
9/14/9

3. $f(x) = \begin{vmatrix} x & x^2 \\ x^4 & x^3 \end{vmatrix}$ Determine $f'(x)$.

$f(x) = x^4 - x^6$; $f'(x) = 4x^3 - 6x^5$

Answer B

4. Determine the Laplace Transform of a unit step function of magnitude a.



$$F(s) = \int_0^\infty f(t)e^{-st}dt = a\int_0^\infty e^{-st}dt =$$

$$= -\frac{a}{s}e^{-st}\Big|_{t=0}^{t=\infty} = -\frac{a}{s}(0-1) = \frac{a}{s}$$

Or from table on page 17 - column 2.

Answer B

30

JAN
9/14/97

5. One wishes to estimate the mean M of a population from a sample of size n drawn from the population. For the sample, the mean is x and the standard deviation is s. The probable accuracy of the estimate improves with increases in:

(A) M
(B) n
(C) s
(D) $M + s$

6. If the functional form of a curve is known, differentiation can be used to determine all of the following EXCEPT the

(A) concavity of the curve.
(B) location of inflection points on the curve.
(C) number of inflection points on the curve.
(D) area under the curve between certain bounds.

7. $\frac{dy}{dt} + 5y = 0;\ y(0) = 1$

Which of the following is the general solution to the differential equation and boundary condition shown above?

(A) e^{5t}
(B) e^{-5t}
(C) $e^{\sqrt{-5t}}$
(D) $5\,e^{-5t}$

8. If D is the differential operator, then the general solution to $(D + 2)^2 y = 0$ is

(A) $C_1 e^{-4x}$

(B) $C_1 e^{-2x}$

(C) $e^{-4x}\,(C_1 + C_2 x)$

(D) $e^{-2x}\,(C_1 + C_2 x)$

5. Desired, estimate of mean M of a populatio
by drawing a sample of size n and
calculating its mean, $\bar{x}$, and its standarc
deviation S.
The probable accuracy of the estimate
improves with ?

Answer B The size of the sample, n.

6. The functional form of a curve is
given. Differentiation can be used to
determine all of the following <u>EXCEPT</u>.

Answer D The area under the curve between certain bounds

32.

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7. Given $\frac{dy}{dt} + 5y = 0$; $y(0) = 1$

Determine $y(t)$

$(D+5)y = 0$; $D = -5$ $y = c_1 e^{-5t}$

$y(0) = 1 = c_1$ $y = e^{-5t}$ Answer B

8. Given $(D+2)^2 y = 0$. Determine $y(x)$.

$D = -2, -2$ - repeated roots.

$y(x) = (c_1 + c_2 x) e^{-2x}$ Answer D

9. Rectilinear Motion $S = 20t^3 - t^4$

Determine da/dt when $t = 2$.

$a = \frac{d^2 s}{dt^2}$. $\frac{ds}{dt} = 60t^2 - 4t^3$; $\frac{d^2 s}{dt^2} = 120t - 12t^2$

$\frac{da}{dt} = 120 - 24t$; When $t = 2$, $\frac{da}{dt} = 120 - 48 = 72$

Answer A

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9/14/97

33

9. A particle travelled in a straight line in such a way that its distance S from a given point on that line after time t was $S = 20t^3 - t^4$. The rate of change of acceleration at time $t = 2$ is:

(A) 72
(B) 144
(C) 192
(D) 208

10. Which of the following is a unit vector perpendicular to the plane determined by the vectors $\mathbf{A} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{B} = \mathbf{i} + \mathbf{j} - \mathbf{k}$?

(A) $-2\mathbf{i} + \mathbf{j} - \mathbf{k}$

(B) $\frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$

(C) $\frac{1}{\sqrt{6}}(-2\mathbf{i} + \mathbf{j} - \mathbf{k})$

(D) $\frac{1}{\sqrt{6}}(-2\mathbf{i} - \mathbf{j} - \mathbf{k})$

11. If f' denotes the derivative of a function of $y = f(x)$, then $f'(x)$ is defined by:

(A) $\lim\limits_{\Delta y \to \infty} \frac{\Delta x}{\Delta y}$

(B) $\lim\limits_{\Delta y \to 0} \frac{\Delta y}{\Delta x}$

(C) $\lim\limits_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

(D) $\lim\limits_{\Delta y \to 0} \frac{f(x - \Delta x) + f(x)}{\Delta y}$

12. What is the area of the region in the first quadrant that is bounded by the line $y = 1$, the curve $x = y^{3/2}$, and the y-axis?

(A) 2/5
(B) 3/5
(C) 2/3
(D) 1

GO ON TO THE NEXT PAGE.

10. Given: $\underline{A} = 2\underline{i} + 4\underline{j}$; $\underline{B} = \underline{i} + \underline{j} - \underline{k}$

Determine Unit Vector $\perp$ plane of $\underline{A}$ & $\underline{B}$.

Vector $\perp$ $\underline{A}, \underline{B}$ is $\underline{A} \times \underline{B} = \begin{vmatrix} i & j & k \\ 2 & 4 & 0 \\ 1 & 1 & -1 \end{vmatrix} = \begin{matrix} -4\underline{i} \\ +2\underline{j} \\ +(2-4)\underline{k} \end{matrix}$

$$\underline{A} \times \underline{B} = -4\underline{i} + 2\underline{j} - 2\underline{k}$$

$$|\underline{A} \times \underline{B}| = \sqrt{4^2 + 2^2 + (-2)^2} = \sqrt{24} = 2\sqrt{6}$$

$$\underline{n} = \frac{1}{2\sqrt{6}}\left(-4\underline{i} + 2\underline{j} - 2\underline{k}\right) = \frac{1}{\sqrt{6}}\left(-2\underline{i} + \underline{j} - \underline{k}\right)$$

Answer C

11. Given $y = f(x)$.

Determine: Definition of f'

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Answer C

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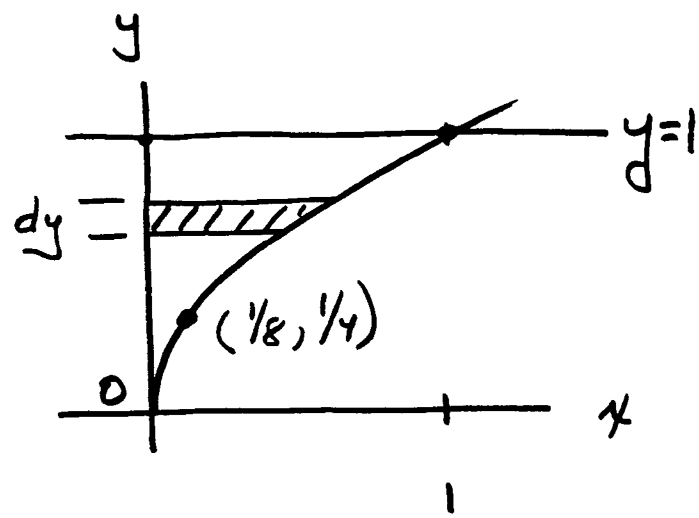
12. Given $x = y^{3/2}$

Determine: Area in First Quadrant Bounded by $y=1$, the curve, and the y-axis



Note: When $x=0, y=0$

$x=1, y=1$

When $y = 1/4$

$x = \left(\frac{1}{4}\right)^{3/2} = \frac{1}{8}$

$$dA = x\,dy = y^{3/2}\,dy$$

$$A = \int_0^1 y^{3/2}\,dy = \frac{2}{5} y^{5/2} \Big|_0^1 = \frac{2}{5}(1-0) = \frac{2}{5}$$

Note: By shape of curve, $A < 1/2$.

Answer A

Only one potential answer was less than $1/2$, namely A

JW
9/14/97

General

30 Questions

Directions: Each of the questions or incomplete statemetns below is followed by four suggested answers or completions. Select the one that is best in each case and then fill in the corresponding space on the answer sheet.

Questions 1-3

Under certain conditions, the motion of an oscillating spring and mass is described by the differential equation

$$\frac{d^2x}{dt^2} + 16x = 0$$

where x is the displacement in feet of the end of the spring, and t is the time in seconds. At $t = 0$ seconds, the displacement is 0.08 m and the velocity is 0 m per second; that is

$$x(0) = 0.08 \text{ and } x'(0) = 0.$$

1. The solution that fits the initial conditions is:

(A) $x = -\sin 4t$
(B) $x = 0.02 \sin 4t + 0.08 \cos 4t$
(C) $x = 4 \cos 4t$
(D) $x = 0.08 \cos 4t$

2. The maximum amplitude of the motion is:

(A) 0.02 m
(B) 0.08 m
(C) 0.16 m
(D) 0.32 m

3. The period of motion is:

(A) $\pi/2$ sec
(B) π sec
(C) 2π sec
(D) 3π sec

37

Mathematics - Afternoon Session - Page 1

Given: $\frac{d^2x}{dt^2} + 16x = 0$; When $t=0$, $x = 0.08\,m$

$\frac{dx}{dt} = 0$

1. Determine $x(t)$.

$(D^2+16)x = 0$; $D = +4i, -4i$, $i = \sqrt{-1}$

$x = C_1 \cos 4t + C_2 \sin 4t$; $x(0) = C_1 = 0.08$

$\dot{x} = -4C_1 \sin 4t + 4C_2 \cos 4t$; $\dot{x}(0) = C_2 = 0$

$x(t) = 0.08 \cos 4t$ Answer D

2. Maximum Amplitude? $x_{MAX} = 0.08\ m$

Answer B

3. Determine the Period.

When the argument of the cosine has reached 2π, the motion has completed one cycle. $4T = 2\pi$

$T = \pi/2$ sec. Answer A

JW 38

9/14/97

4. The equation of the line normal to the curve defined by the function $y(x) = 8 - 2x^2$ at the point (1,6) is:

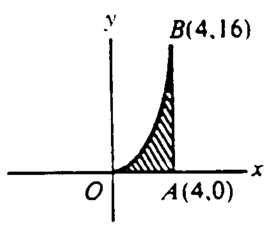
(A) $\frac{y-6}{x-1} = 4$

(B) $\frac{y-6}{x-1} = \frac{1}{4}$

(C) $\frac{y-6}{x-1} = -4$

(D) $\frac{y+6}{x+1} = -4$

Questions 5-6 pertain to the area OAB bounded by $y = x^2$, $y = 0$, and $x = 4$ as shown below.



5. What is the area of OAB?

(A) 64/3
(B) 32
(C) 128/3
(D) 256/3

6. What is the first moment of the area OAB about the x-axis?

(A) 32
(B) 64
(C) 512/5
(D) 1,024/5

4. Given $y = 8 - 2x^2$. At the point $(1, 6)$ Determine the equation of the line normal to the curve.

The slope of the curve $= y' = -4x$.

At point $x = 1$, $y = 6$, slope $= -4 = y'(1)$

Slope of normal is the negative of the reciprocal or $+1/4$. Thus

$$y_n = mx + b = \frac{1}{4}x + b.$$

When $x = 1$, $y_n = 6$

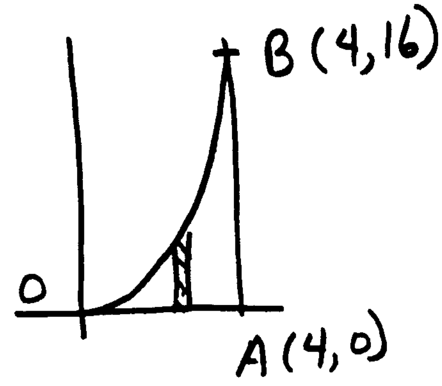
$$6 = \frac{1}{4}x + b \; ; \; b = 6 - \frac{1}{4}$$

$$y = \frac{1}{4} \quad + 6 - \frac{1}{4} ; \quad y - 6 = \frac{1}{4}x - \frac{1}{4} = \frac{x-1}{4}$$

$$\frac{y-6}{x-1} = \frac{1}{4}$$

Answer B

Given the Area OAB bounded by $y = x^2$; $y = 0$ and $x = 4$



5. Determine Area OAB

$$dA = y\,dx = x^2\,dx$$

$$A = \int_0^4 x^2\,dx = \frac{1}{3}x^3\Big|_0^4 = \frac{1}{3}(64-0) = 64/3$$

Answer (A)

6. Determine First Moment of OAB about the x-axis.

$$dM_x = \frac{1}{2}y\,dA = \frac{1}{2}y^2\,dx = \frac{1}{2}x^4\,dx$$

$$M_x = \frac{1}{2}\int_0^4 x^4\,dx = \frac{1}{2}\cdot\frac{1}{5}x^5\Big|_0^4 = \frac{1}{10}(1024-0)$$

$$M_x = \frac{512}{5}$$

Answer (C)

JW 41

9/14/97

Diagnostic Examination

TOPIC III: MATHEMATICS

TIME LIMIT: 45 MINUTES

1. What is the general form of the equation for a line whose x-intercept is 4 and y-intercept is -6?

(A) $2x - 3y - 18 = 0$
(B) $2x + 3y + 18 = 0$
(C) $3x - 2y - 12 = 0$
(D) $3x + 2y + 12 = 0$

#4496 895

2. For some angle θ, $\csc\theta = -8/5$. What is $\cos 2\theta$?

(A) 7/32
(B) 1/4
(C) 3/8
(D) 5/8

#4368 895

3. What is the rectangular form of the following polar equation?

$$r^2 = 1 - \tan^2\theta$$

(A) $-x^2 + x^4y^2 + y^2 = 0$
(B) $x^2 + x^2y^2 - y^2 + y^4 = 0$
(C) $-x^4 + y^2 = 0$
(D) $x^4 - x^2 + x^2y^2 + y^2 = 0$

#4387 1295

4. For three matrices **A**, **B**, and **C**, which of the following statements is not necessarily true?

(A) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
(B) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
(C) $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{AB} + \mathbf{AC}$
(D) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = \mathbf{C} + (\mathbf{A} + \mathbf{B})$

#4320 196

5. For the three vectors **A**, **B**, and **C**, what is the product $\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C})$?

$$\mathbf{A} = 6\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}$$
$$\mathbf{B} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$
$$\mathbf{C} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$$

(A) 0
(B) 64
(C) 80
(D) 216

#4556 796

6. The second and sixth terms of a geometric progression are 3/10 and 243/160, respectively. What is the first term of this sequence?

(A) 1/10
(B) 1/5
(C) 3/5
(D) 3/2

#4397 1295

7. A marksman can hit a bull's-eye from 100 m with three out of every four shots. What is the probability that he will hit a bull's-eye with at least one of his next three shots?

(A) 3/4
(B) 15/16
(C) 31/32
(D) 63/64

#4230 596

8. The final scores of students in a graduate course are distributed normally with a mean of 73 and a standard deviation of 11. What is the probability that a student's score will be between 65 and 80?

(A) 0.4196
(B) 0.4837
(C) 0.5161
(D) 0.6455

#4401 1295

9. Evaluate the following limit.

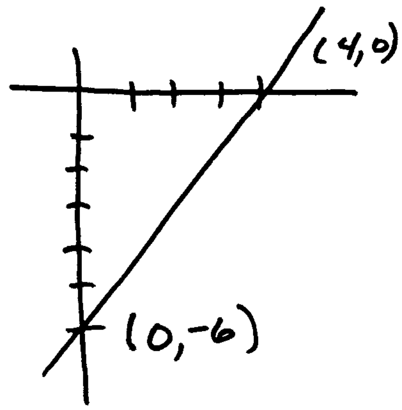
$$\lim_{x\to 0}\left(\frac{1 - e^{3x}}{4x}\right)$$

(A) $-\infty$
(B) $-3/4$
(C) 0
(D) 1/4

#4423 596

4

1. General form for a line with x intercept of 4, y intercept of -6.



Only (C) Works

$3x - 2y - 12 = 0$

2. Given that $\csc\theta = -8/5$, what is $\cos 2\theta$

$\csc\theta = \frac{1}{\sin\theta}$, $\sin\theta = -\frac{5}{8}$

$\theta = \arcsin(-5/8) = -38.68$

$\cos(2 \times -38.68) = +0.21875$

$32 \times 0.21875 = 7.$ (A)

3. Rectangular Form for

$$r^2 = 1 - \tan^2\theta$$

$$r^2 = x^2 + y^2 = 1 - \tan^2\theta = 1 - \frac{y^2}{x^2}$$

$$x^4 + x^2y^2 = x^2 - y^2$$

$$x^4 - x^2 + x^2y^2 + y^2 = 0 \qquad \text{D}$$

4. Matrices A, B, C. - Which is not necessarily true?

$\cancel{A \neq}$ C $(B+C)A = AB + AC.$

5. $\underline{A} = 6\underline{i} + 8\underline{j} + 10k$

$\underline{B} = \underline{i} + 2\underline{j} + 3\underline{k}$ $\qquad \underline{A} \cdot (\underline{B} \times \underline{C}) = ?$

$\underline{C} = 3\underline{i} + 4\underline{j} + 5\underline{k}$

$$\underline{B} \times \underline{C} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = \begin{matrix} \underline{i}\,(2\times5 - 3\times4) \\ -\underline{j}\,(1\times5 - 3\times3) \\ +\underline{k}\,(1\times4 - 2\times3) \end{matrix} = \begin{matrix} -2\underline{i} \\ +4\underline{j} \\ -2\underline{k} \end{matrix}$$

4.4

$$(6\underline{i} + 8\underline{j} + 10\underline{k}) \cdot (-2\underline{i} + 4\underline{j} - 2\underline{k}) = -12 + 32 - 20 = 0 \qquad \text{A}$$

7. Marksman scores on 3 of 4 shots.

$P(H \geq 1)$ with next 3 shots.

$P(H) = 3/4, \quad P(\text{Miss}) = 1/4.$

$$P(\text{Misses} = 3) = [P(\text{Miss})]^3 = \frac{1}{64}$$

$$P(H \geq 1) = 1 - 1/64 = 63/64 \quad \text{D}$$

6. Geometric Progression $T_2 = 3/10$, $T_6 = 243/160$

$T_1 = ?$

$$T_2 = T_1 \, r$$

$$T_6 = T_5 r = T_4 r^2 = T_3 r^3 = T_2 r^4$$

$$\frac{243}{160} = \frac{3}{10} r^4 ; \quad r^4 = \frac{243}{160} \times \frac{10}{3} = \frac{81}{16}$$

$$r = \frac{3}{2}$$

$$T_2 = \frac{3}{10} = T_1 \left(\frac{3}{2}\right); \quad T_1 = \frac{3}{10} \cdot \frac{2}{3} = 1/5 \quad \text{B}$$

8. Normally Distributed Scores.

$\text{Mean} = 73, \quad \sigma = 11$

$P(65 < \text{Score} < 80) = ? \qquad z = \frac{s - \mu}{\sigma}$

$z_1 = \frac{-73+65}{11} = -8/11 \approx 0.73 \approx 0.7.$

$z_2 = \frac{80-73}{11} = 7/11 \approx 0.64 \approx 0.6.$

$P(65 \le \text{Score} \le 80) = F(.60) - R(0.70)$

$= 0.7257 - 0.2420$

$= 0.4837$

B

9. Evaluate $\lim_{x \to 0} \left(\frac{1 - e^{3x}}{4x} \right)$ $\qquad \frac{0}{0}$

$$= \lim_{x \to 0} \left(\frac{-3e^{3x}}{4} \right) = -3/4$$

B

10. Snowball, $dr/dt = 20$ cm/min

What is dV/dt when $D = 1$ m.?

When $D = 1$ m, $r = 0.5$ m

$$V = \frac{4}{3}\pi r^3; \quad \frac{dV}{dt} = 3 \cdot \frac{4}{3}\pi r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi \left(\frac{1}{2}\right)^2 \left(\frac{20}{100}\right) = 0.2\pi = 0.63 \text{ m}^3/\text{min}$$

C

10. The radius of a snowball rolling down a hill is increasing at a rate of 20 cm/min. How fast is its volume increasing when its diameter is 1 m?

(A) 0.034 m^3/min
(B) 0.52 m^3/min
(C) 0.63 m^3/min
(D) 0.84 m^3/min

#4463 995

11. Evaluate the following indefinite integral.

$$\int \cos^2 x \sin x \, dx$$

(A) $-\frac{2}{3}\sin^3 x + C$

(B) $-\frac{1}{3}\cos^3 x + C$

(C) $\frac{1}{3}\sin^3 x + C$

(D) $\frac{1}{2}\sin^2 x \cos^2 x + C$

#4159 694

12. What is the area of the region bounded by the curve $y = \sin x$ and the x-axis on the interval between $x = \pi/2$ and $x = 2\pi$?

(A) 1
(B) 2
(C) 3
(D) 4

#4558 796

13. What is the general solution to the following differential equation?

$$y'' - 8y' + 16y = 0$$

(A) $y = C_1 e^{4x}$
(B) $y = (C_1 + C_2 x)e^{4x}$
(C) $y = C_1 e^{-4x} + C_2 e^{4x}$
(D) $y = C_1 e^{2x} + C_2 e^{4x}$

#4050 495

14. Find the Laplace transform of the equation with the given boundary conditions.

$$f''(t) + f(t) = \sin \beta t$$
$$f'(0) = 0$$
$$f(0) = 0$$

(A) $F(s) = \left(\frac{1}{1+s^2}\right)\left(\frac{\beta}{s^2+\beta^2}\right)$

(B) $F(s) = \left(\frac{1}{1+s^2}\right)\left(\frac{\beta}{s^2-\beta^2}\right)$

(C) $F(s) = \left(\frac{1}{1-s^2}\right)\left(\frac{\beta}{s^2+\beta^2}\right)$

(D) $F(s) = \left(\frac{1}{1-s^2}\right)\left(\frac{s}{s^2+\beta^2}\right)$

#4428 596

15. Newton's method is being used to find the roots of the equation $f(x) = (x-2)^2 - 1$. What is the third approximation of the root if 9.33 is chosen as the first approximation?

(A) 1.0
(B) 2.0
(C) 3.0
(D) 4.0

#4557 796

SOLUTIONS TO DIAGNOSTIC EXAMINATION TOPIC III

Solution 1:

Find the slope of the line.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-6-0}{0-4} \\ &= 3/2 \end{aligned}$$

Once the slope and y-intercept are known, the slope-intercept form is convenient to use.

$$y = mx + b$$
$$mx - y + b = 0$$
$$\frac{3}{2}x - y + (-6) = 0$$
$$3x - 2y - 12 = 0$$

48

Answer is C.

11. Evaluate $\int \cos^2 x \sin x \, dx$

Let $u = \cos x$ $\quad du = -\sin x \, dx$

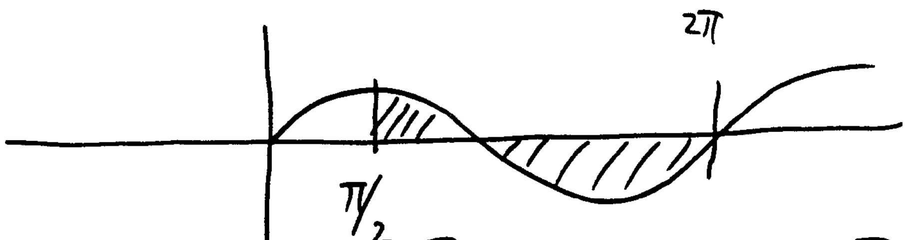
$$\int \cos^2 x \sin x \, dx = -\int \cos^2 x \, (-\sin x \, dx)$$

$$= -\int u^2 \, du = -\frac{1}{3} u^3 + C$$

$$= -\frac{\cos^3 x}{3} + C$$

B

12. Area Bounded by $y = \sin x$, x-axis, $x = \pi/2$ and $x = 2\pi$



$$A = \int_{\pi/2}^{\pi} \sin x \, dx - \int_{\pi}^{2\pi} \sin x \, dx = -\cos x \Big|_{\pi/2}^{\pi} + \cos x \Big|_{\pi}^{2\pi}$$

$$= -\cos \pi + \cos \pi/2 + \cos 2\pi - \cos \pi = 3$$

C

13. DE. $y'' - 8y' + 16y = 0$

$$y = ce^{rx}$$

$$r^2 - 8r + 16 = (r-4)(r-4) = 0$$

$$r = +4, +4$$

$$y = c_1 e^{4x} + c_2 x e^{4x}$$

B

14. DE $f''(t) + f(t) = \sin\beta t \quad f'(0) = f(0) = 0$

Determine $\mathcal{L}\{f(t)\} = F(s)$

$$s^2 F(s) - s \cdot 0 - 0 + F(s) = \frac{\beta}{s^2+\beta^2}$$

p17

$$F(s) = \frac{1}{1+s^2} \cdot \frac{\beta}{s^2+\beta^2}$$

A

15. $f(x) = (x-2)^2 - 1$

Newton's method to extract roots.

What is x_3 if $x_1 = 9.33$?

Newton's Method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$f'(x) = 2(x-2)$

$f(x_1) = (9.33-2)^2 - 1 = 52.73$; $f'(x_1) = 2(9.33-2) = 14.66$

$x_2 = 9.33 - \dfrac{52.73}{14.66} = 9.33 - 3.60 = 5.73$

$f(x_2) = (5.73-2)^2 - 1 = 12.92$; $f'(x_2) = 2(5.73-2) = 7.46$

$x_3 = 5.73 - \dfrac{12.92}{7.46} = 4.0$

19. What is the train's velocity at time $t=4$ hours?

(A) −16 km/h (−16 miles/hr)
(B) −8 km/h (−8 miles/hr)
(C) 0 km/h (0 miles/hr)
(D) 32 km/h (32 miles/hr)
(E) 64 km/h (64 miles/hr)

ADDITIONAL PRACTICE PROBLEMS

20. What is the train's acceleration at time $t=4$ hours?

(A) −16 km/h² (−16 miles/hr²)
(B) 0 km/h² (0 miles/hr²)
(C) 12 km/h² (12 miles/hr²)
(D) 16 km/h² (16 miles/hr²)
(E) 32 km/h² (32 miles/hr²)

Questions 21-24

Under certain conditions, the motion of an oscillating spring and mass is described by the differential equation

$$\frac{d^2x}{dt^2}+16x=0$$

where x is the displacement in feet of the end of the spring, and t is the time in seconds. At $t=0$ seconds, the displacement is $\frac{1}{4}$ foot and the velocity is 0 feet per second; that is, $x(0)=\frac{1}{4}$ and $x'(0)=0$.

MATHEMATICS

21. What is the general solution of the system? (c_1 and c_2 are constants.)

(A) $x=c_1e^{t}+c_2e^{-4t}$
(B) $x=c_1e^{-4t}+c_2e^{4t}$
(C) $x=c_1 \sin 4t$
(D) $x=c_1 \cos 4t$
(E) $x=c_1 \cos 4t+c_2 \sin 4t$

22. The solution that fits the initial conditions is:

(A) $x=\frac{1}{4}e^{-4t}$

(B) $x=\frac{1}{3}\sin 4t$

(C) $x=\frac{1}{3}\sin 4t+\frac{1}{4}\cos 4t$

(D) $x=4\cos 4t$

(E) $x=\frac{1}{4}\cos 4t$

23. The amplitude of the motion is:

(A) $\frac{1}{4}$ ft

(B) $\frac{1}{3}$ ft

(C) 1 ft
(D) 2 ft
(E) 4 ft

52

GO ON TO THE NEXT PAGE.

24. The period of the motion is:

(A) $\frac{\pi}{3}$ sec

(B) $\frac{\pi}{2}$ sec

(C) π sec

(D) 2π sec

(E) 3π sec

Questions 25–26 relate to the curve defined by the function $y(x) = 8-2x^2$ in Cartesian coordinates.

25. The line tangent to the curve at the point (1,6) intersects the x-axis at:

(A) −23

(B) 2

(C) $\frac{7}{3}$

(D) $\frac{5}{2}$

(E) 10

26. The equation of the line normal to the curve at the point (1,6) is:

(A) $\frac{y-6}{x-1} = 4$

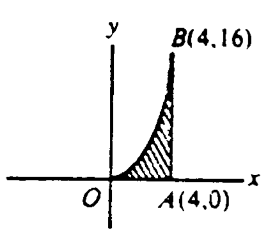
(B) $\frac{y-6}{x-1} = \frac{1}{4}$

(C) $\frac{y-6}{x-1} = -4$

(D) $\frac{y+6}{x+1} = -4$

(E) $\frac{x-1}{y-6} = -\frac{1}{4}$

Questions 27–29 pertain to the area OAB bounded by $y=x^2$, $y=0$, and $x=4$, as shown below.



53

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27. What is the area OAB?

(A) $\frac{3}{64}$

(B) 8

(C) $\frac{64}{3}$

(D) $\frac{43}{2}$

(E) 32

28. What is the first moment of the area OAB about the x-axis?

(A) 4

(B) $\frac{24}{5}$

(C) $\frac{64}{3}$

(D) 101

(E) $\frac{512}{5}$

29. What volume is generated by revolving the area OAB around the x-axis?

(A) $\frac{512\pi}{5}$

(B) $\frac{512\pi}{3}$

(C) 200π

(D) $\frac{1024\pi}{5}$

(E) $\frac{1024\pi}{3}$

Questions 30–34 relate to the function $y=2x^3-3x^2-36x+25$.

30. What is the slope of the line tangent to the graph of this function at the point $(1,-12)$?

(A) −42
(B) −36
(C) −17
(D) 28
(E) 36

31. At how many points will the line tangent to the graph of this function be horizontal?

(A) At no point
(B) At exactly one point
(C) At exactly two points
(D) At exactly three points
(E) At exactly four points

54

GO ON TO THE NEXT PAG

32. The function assumes a relative minimum at what value of x?

(A) −1
(B) 0
(C) 1
(D) 2
(E) 3

33. The graph of this function has an inflection point at:

(A) x=−2
(B) x= 0
(C) x= $\frac{1}{2}$
(D) x= 3
(E) no value of x.

34. The value of $y'''(1)$, the third derivative of y with respect to x at x=1, is:

(A) −6
(B) 0
(C) 6
(D) 12
(E) undefined

Questions 35-37

The motion of an oscillating spring and mass, under certain conditions, is described by the differential equation

$$\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=0,$$

where x is the displacement in meters of the end of the spring at time t in seconds. The initial displacement $x(0)$ is $\frac{1}{2}$ meter and the initial velocity $x'(0)$ is 0.

35. What is the general solution of the differential equation? (c_1 and c_2 are constants.)

(A) $x=c_1e^{2t}+c_2e^{3t}$
(B) $x=c_1e^{-2t}+c_2e^{-3t}$
(C) $x=c_1e^{2t}+c_2e^{-3t}$
(D) $x=c_1e^{-2t}+c_2e^{3t}$
(E) $x=c_1e^{-5t}$

36. The solution that fits the initial conditions is:

(A) $x=\frac{1}{2}e^{-2t}$

(B) $x=\frac{1}{2}e^{-3t}$

(C) $x=-e^{-2t}+\frac{3}{2}e^{-3t}$

(D) $x=e^{-2t}+\frac{1}{2}e^{-3t}$

(E) $x=\frac{3}{2}e^{-2t}-e^{-3t}$

GO ON TO THE NEXT PAGE.

37. The displacement at $t=1$ is:

(A) $e^{-2}\left(1-e^{-1}\right)$

(B) $e^{-2}\left(\frac{3}{2}-e^{-1}\right)$

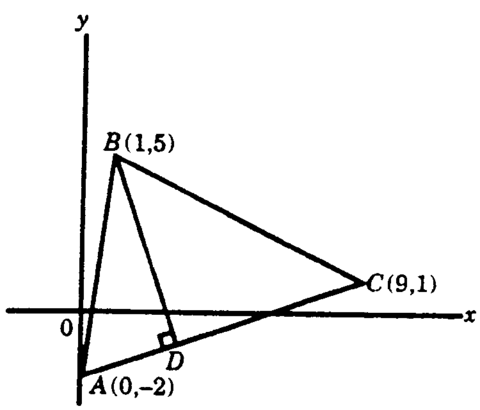
(C) $e^{-2}\left(1-\frac{3}{2}e^{-1}\right)$

(D) $e^{2}\left(1-e^{-1}\right)$

(E) $e^{2}\left(\frac{3}{2}-e^{-1}\right)$

Questions 38–40

Triangle ABC has vertices as shown in the figure below. The line BD is perpendicular to the line AC.



38. What is the length of line AC?

(A) $2\sqrt{3}$
(B) $3\sqrt{10}$
(C) 10
(D) 11
(E) 12

39. What is the equation of line AC?

(A) $y=\frac{1}{3}x-2$

(B) $y=-\frac{1}{3}x+2$

(C) $x+3y=6$
(D) $x-3y+6=0$
(E) $x+y=6$

40. What is the slope of line BD?

(A) –5
(B) –3
(C) $-\frac{1}{3}$
(D) $\frac{1}{3}$
(E) 3

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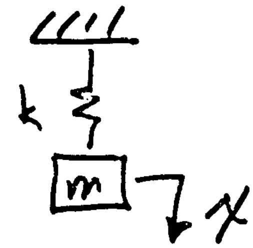
FE Refresher



Spring/Mass Oscillation

$$\frac{d^2x}{dt^2} + 16x = 0 \qquad x(0) = \frac{1}{4}\text{ ft} \qquad \dot{x}(0) = 0$$

21. What is general solution?

$$(D^2 + 16)x = 0 \qquad D = \mp 4i$$

$$\boxed{x = C_1 \cos 4t + C_2 \sin 4t \quad \text{Solution E)}}$$

22. What is $x(t)$?

$$\dot{x} = -4C_1 \sin 4t + 4C_2 \cos 4t$$

$$\dot{x}(0) = 4C_2 = 0; \quad \boxed{C_2 = 0}$$

$$x(0) = C_1 \cos(0) = \boxed{C_1 = \frac{1}{4}}$$

$$\boxed{x = \frac{1}{4}\cos 4t \quad \text{Solution E)}}$$

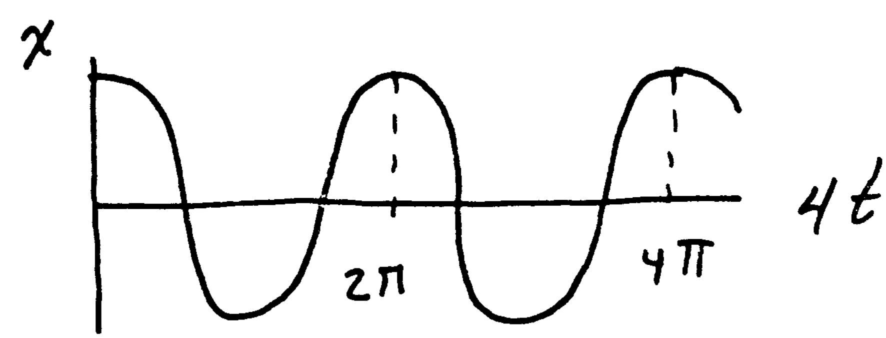
23. What is the Amplitude?

Amplitude = maximum value of x

$$\boxed{x_{max} = \tfrac{1}{4}\text{ft} \quad \text{Solution A)}}$$

24. What is the period, T

T = minimum time for motion to repeat.



$$\frac{1}{4}\cos(4(t+T)) = \frac{1}{4}\cos(4t)$$

$$4T = 2\pi$$

$$\boxed{T = \text{Period} = \frac{\pi}{2} \quad \text{Solution B}}$$

Given $y = 8 - 2x^2$

25. A tangent is drawn at (1, 6). Where does it cross the x-axis?

Note: when $x = 1$, $y = 8 - 2 = 6$

$$y' = \frac{dy}{dx} = -4x$$

When $x = 1$, $y' = -4 =$ slope of tangent

Equation of tangent

$$y_T = mx + b = -4x + b.$$

But when $x = 1$, $y_T = 6$

$$y_T = 6 = -4(1) + b \; ; \; b = 10$$

$$y_T = -4x + 10$$

Intersects x-axis when $y_T = 0$

$$0 = -4x + 10 \; ; \; x = \frac{10}{4}$$

26. Equation normal to curve at (1,6)

Since, at $x=1$, $y' = -4(1) = -4$,

the slope of the normal will

be $-\frac{1}{m}$ or $-\frac{1}{-4} = +\frac{1}{4}$.

Thus

$$y_N = \frac{1}{4}x + b_N$$

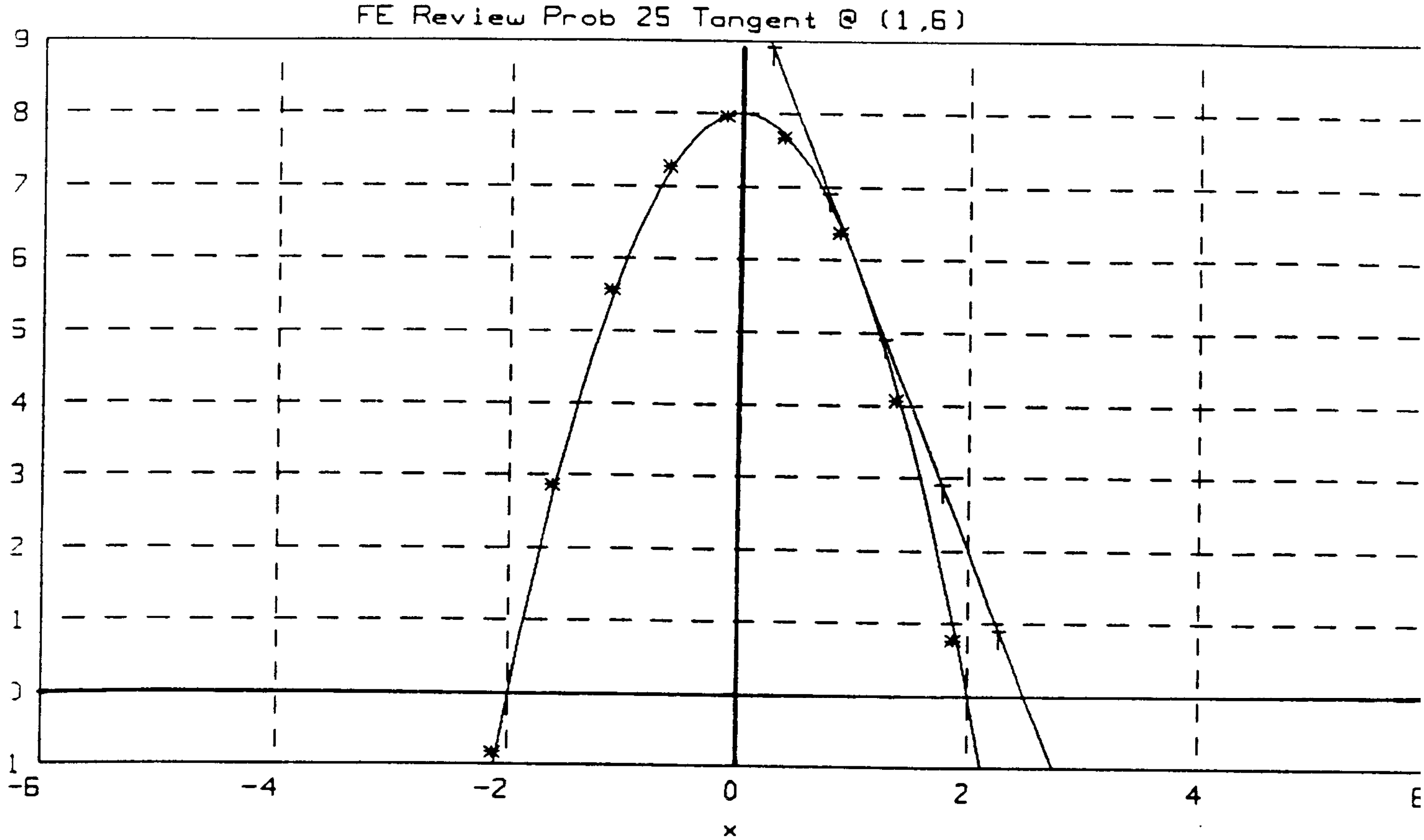
When $x=1$, $y_N = 6 = \frac{1}{4} + b_N$; $b_N = 6 - \frac{1}{4}$

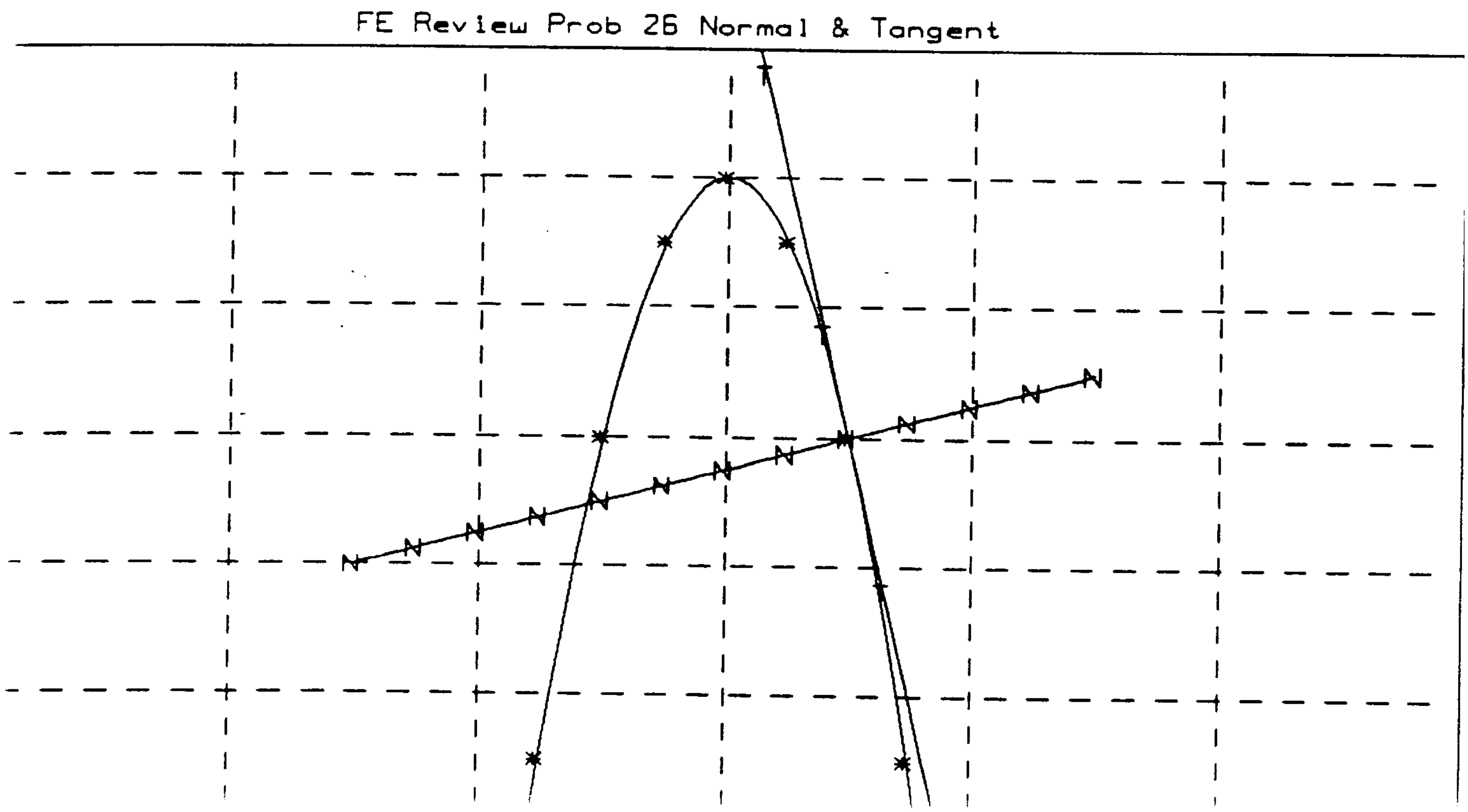
$$y_N = \frac{1}{4}x + 6 - \frac{1}{4} = \frac{x-1}{4} + 6$$

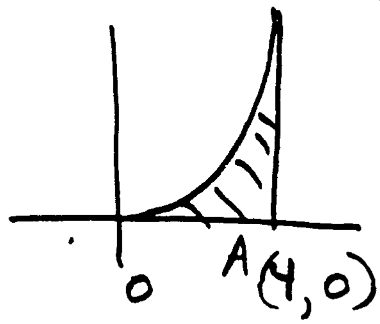
or $$y_N - 6 = \frac{x-1}{4}$$

or $$\boxed{\frac{y_N - 6}{x-1} = \frac{1}{4} \quad \text{Solution B)}}$$

4a


FE Review Prob 25 Tangent @ (1,6)
9
8
7
6
5
4
3
2
1
0
-1
-6
-4
-2
0
2
4
6
x


FE Review Prob 26 Normal & Tangent

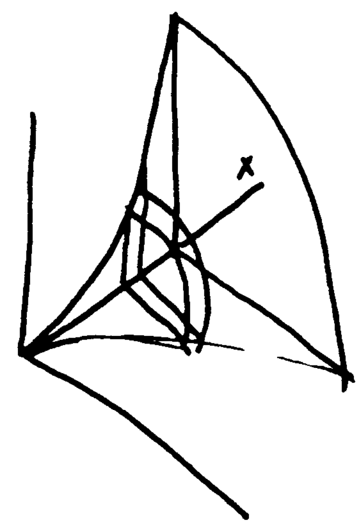


$y = x^2$

27. Area $OAB = \int_0^4 y\,dx = \int_0^4 x^2\,dx = \frac{x^3}{3}\Big|_0^4 = \frac{64}{3}$

c) Area $= \frac{64}{3}$

29 Volume Rotated About x-axis



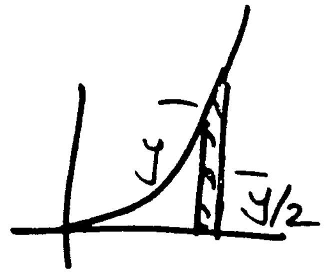
$dv = \pi y^2\,dx$

$dv = \pi (x^2)^2\,dx = \pi x^4\,dx$

$V = \pi \int_0^4 x^4\,dx = \frac{\pi}{5} x^5 \Big|_0^4$

$V = \frac{1024\pi}{5}$ = D)

28. First Moment about x-axis.



$$dM_x = \bar{y}\, dA =$$

$$= \frac{y}{2} \cdot y\, dx$$

$$dM_x = \frac{1}{2} y^2 dx$$

$$M_x = \frac{1}{2}\int_0^4 x^4\, dx = \frac{1}{2}\frac{x^5}{5}\Big|_0^4$$

$$M_x = \frac{1024}{2} \cdot \frac{1}{5} = \frac{512}{5}$$

Ans E)

30-34. $y = 2x^3 - 3x^2 - 36x + 25$

30. Slope of tangent at $(1, -12)$

Note $y(1) = -12$ <u>OK</u>

$y' = 6x^2 - 6x - 36$

$y'(1) = 6 - 6 - 36 = \boxed{-36 = \text{B)}}$

31. At what points will tangent be horizontal?

$y' = 6(x^2 - x - \overset{6}{\cancel{36}}) = 0$

$x = \dfrac{1 \mp \sqrt{1 + \cancel{3}6 \cdot 4}}{2} = \dfrac{1 \mp \overset{5}{\cancel{12.04}}}{2}$ 3, -

slope $= 0$ at $\boxed{\text{2 points} = \text{C)}}$

32. $y = \text{minimum at } x = ?$

$y = \text{min where } y' = 0 \text{ and } y'' > 0$

$y' = 0 \text{ at } x = 3, -2.$

$y'' = 6(2x - 1) =$

$y''(3) = 6(2 \cdot 3 - 1) = 30$

$\therefore\ y_{min} \text{ occurs at } \boxed{x = 3 = E)}$

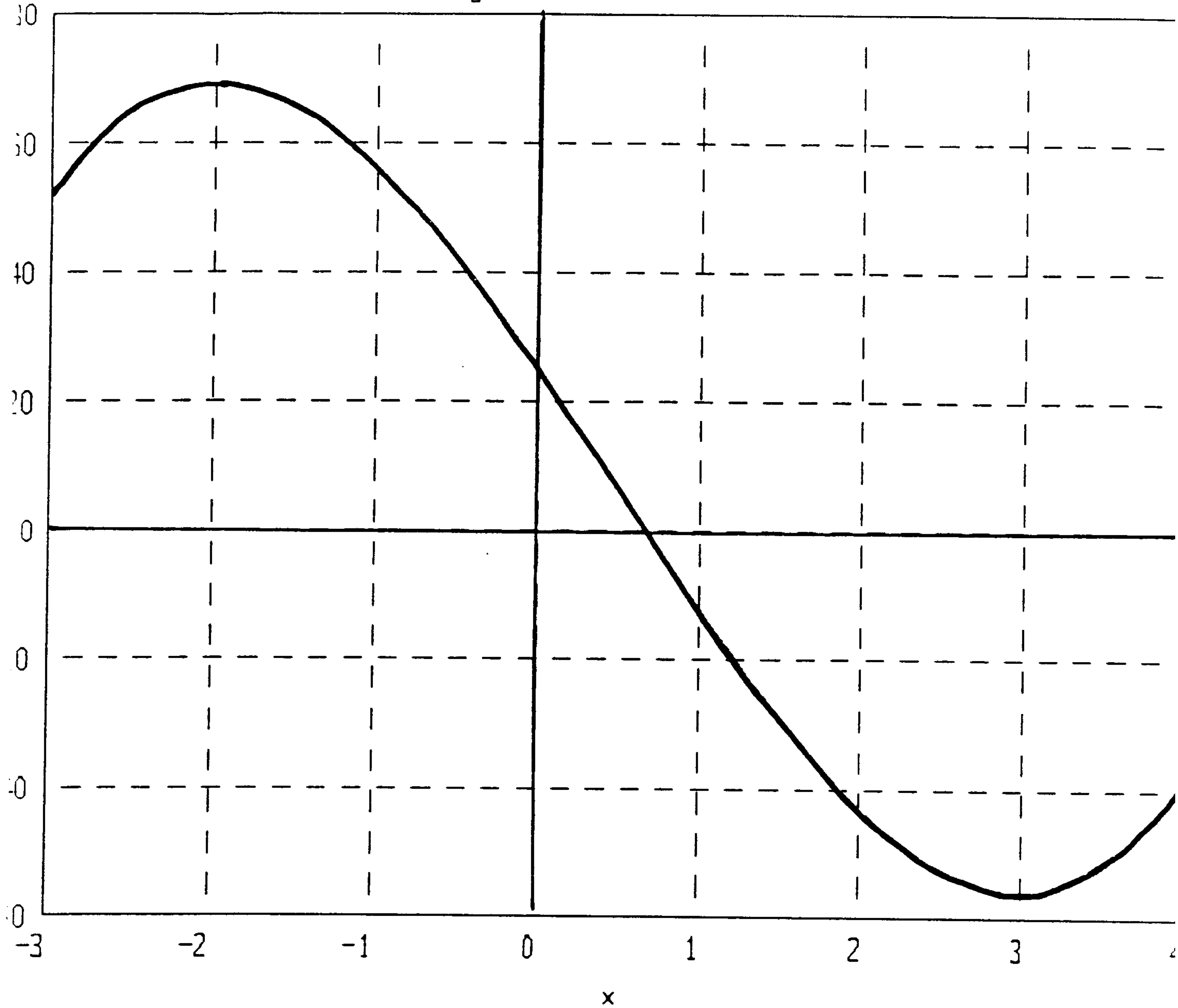
33. Inflection Point Occurs Where $y'' = 0$

$y'' = 6(2x - 1) = 0\ ;\ x = +1/2$

$\boxed{y \text{ inflection at } x = 1/2 = C)}$

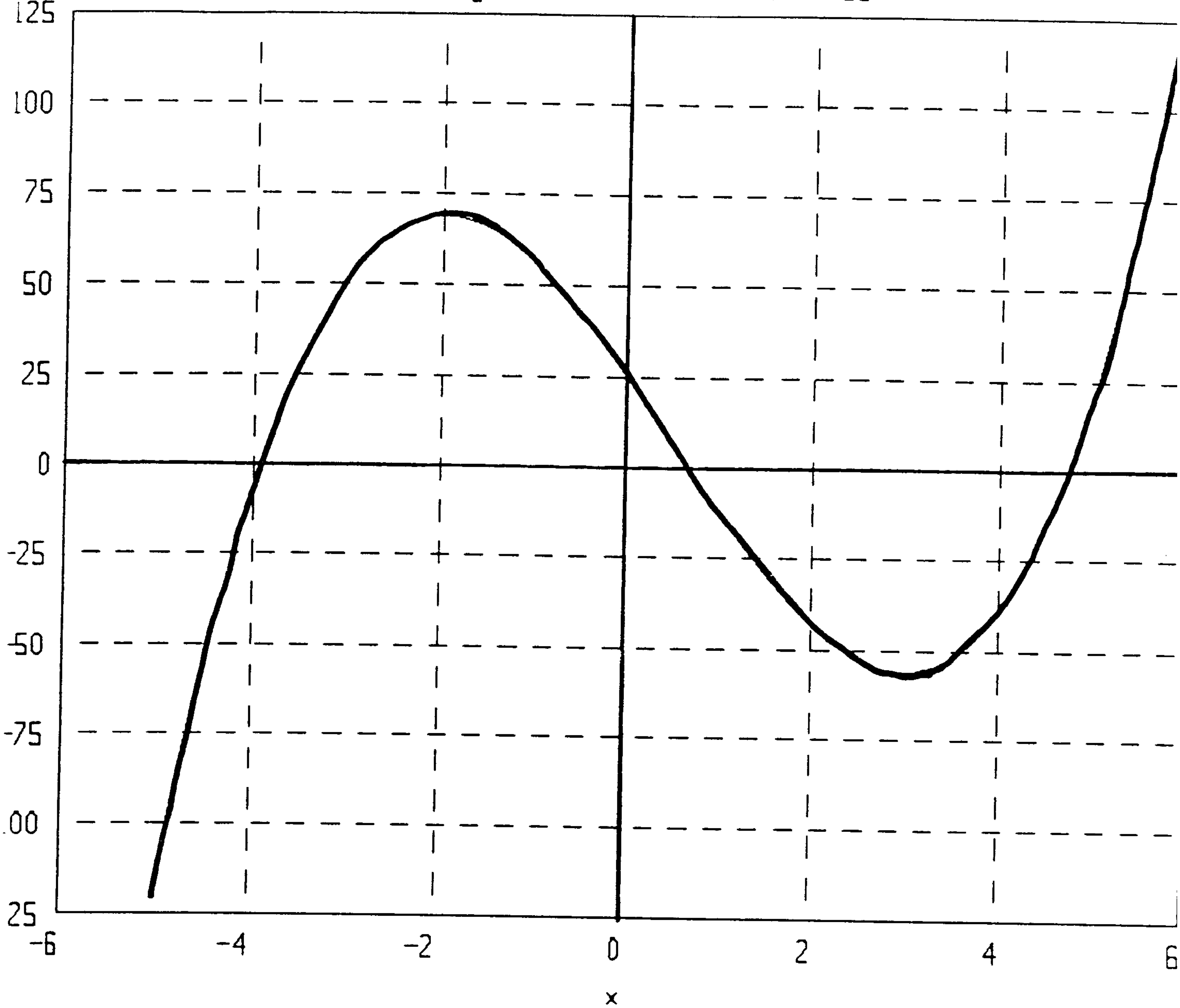
34. $y''' = 12\ ;\ \text{at } x = 1, \boxed{y''' = 12 = D)}$

FE Probs 30 - 34 y = 2x^3 + 3x^2 - 36x + 25



7b

FE Probs 30 - 34 y = 2x^3 + 3x^2 - 36x + 25



$$\ddot{x} + 5\dot{x} + 6x = 0 \qquad x(0) = \frac{1}{2}, \quad \dot{x}(0) = 0$$

35. General Solution.

$$(D^2 + 5D + 6)x = 0;$$

$$(D+2)(D+3)x = 0, \quad D = -2, -3.$$

$$x = C_1 e^{-2t} + C_2 e^{-3t} = B)$$

36. $$\dot{x} = -2C_1 e^{-2t} - 3C_2 e^{-3t}$$

$$\dot{x}(0) = -2C_1 - 3C_2 = 0; \qquad C_1 = -\frac{3}{2}C_2$$

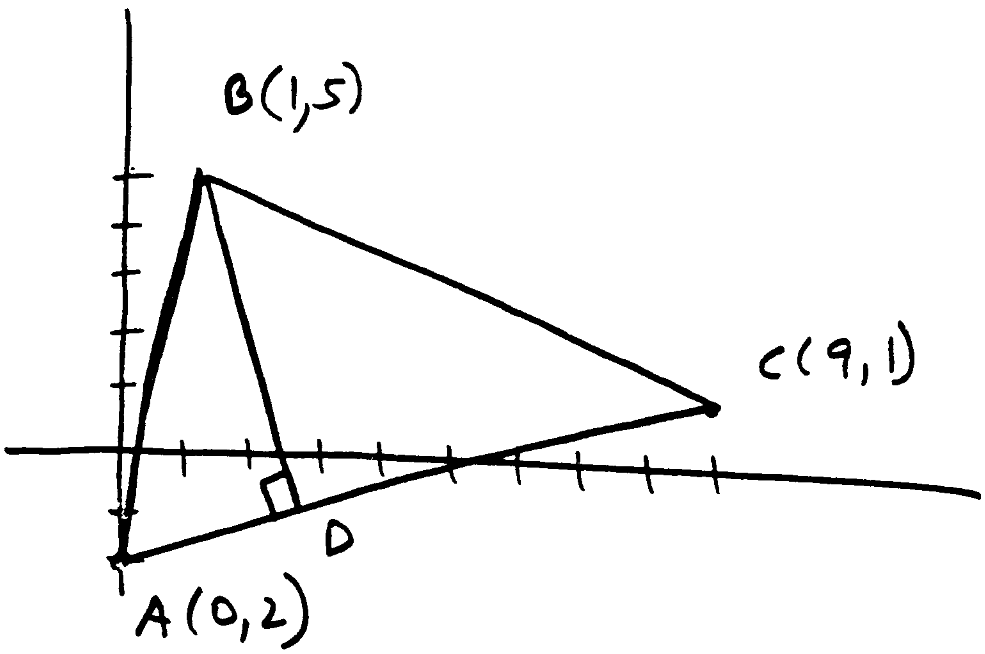
$$x(0) = C_1 + C_2 = -\frac{3}{2}C_2 + C_2 = -\frac{1}{2}C_2 = \frac{1}{2}$$

$$C_2 = -1, \quad C_1 = +\frac{3}{2}$$

$$\boxed{x = \frac{3}{2}e^{-2t} - 1\, e^{-3t} = \underline{E)}}$$

37. When $t=1$

$$x = \frac{3}{2}e^{-2} - e^{-3} = \underline{e^{-2}\left(\frac{3}{2} - e^{-1}\right) = B)}$$



38. What is length of AC

$$L = \sqrt{9^2+3^2} = \sqrt{81+9} = \sqrt{90} = \underline{3\sqrt{10}}$$

B)

39. Equation of AC. $y = mx + b$ $\quad b = -2$

$m = 3/9 = 1/3$

(10)

40. What is slope of BD?

$$m_{BD} = -\frac{1}{m_{AC}} = -\frac{1}{1/3} = \boxed{-3 = B}$$

Two smooth spheres are inserted in a frictionless square slot. Sphere A weighs 10 pounds and sphere B weighs 40 pounds. The angle that the line of contact makes with the horizontal is 30°.

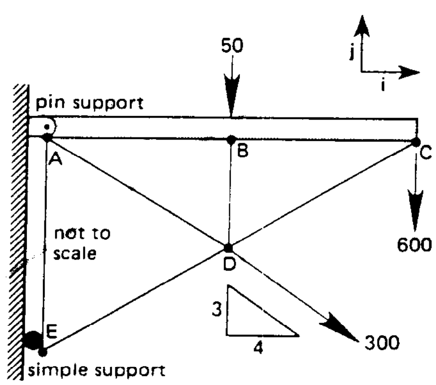
16. What is the normal force, F, at the bottom of the slot?

    (A) 30 lbf (B) 40 lbf (C) 45 lbf
    (D) 50 lbf (E) 57 lbf

17. What is the wall reaction, N_B?

    (A) 5.8 lbf (B) 15 lbf (C) 17.3 lbf
    (D) 20 lbf (E) 35 lbf

Problems 18–20 are based on the following diagram and statement.



dimensions of members

ABC	7½
BD	2
DC	4¼
ED	4¼
AD	4¼

note: ABC is one member
Angles EAC and DBC are 90°

A simple truss mechanism is loaded as shown.

18. What are the reactions at pin A and roller E?

    (A) $\mathbf{A} = -1461\mathbf{i} + 830\mathbf{j}$; $\mathbf{E} = +1276.9\mathbf{i}$
    (B) $\mathbf{A} = -1461\mathbf{i} + 830\mathbf{j}$; $\mathbf{E} = +1221\mathbf{i}$
    (C) $\mathbf{A} = -980\mathbf{i} - 730\mathbf{j}$; $\mathbf{E} = +1389.4\mathbf{i}$
    (D) $\mathbf{A} = +1400\mathbf{i} - 370\mathbf{j}$; $\mathbf{E} = +658.1\mathbf{i}$
    (E) $\mathbf{A} = -1400\mathbf{i} + 730\mathbf{j}$; $\mathbf{E} = +1262.8\mathbf{i}$

19. What are the forces in members AD and ED?

    (A) $F_{AD} = 676$ (tension); $F_{ED} = 346$ (tension)
    (B) $F_{AD} = 676$ (compression); $F_{ED} = 346$ (compression)
    (C) $F_{AD} = 380$ (tension); $F_{ED} = 1383$ (compression)
    (D) $F_{AD} = 380$ (compression); $F_{ED} = 1383$ (tension)
    (E) $F_{AD} = 676$ (tension); $F_{ED} = 692$ (tension)

20. What additional horizontal force applied at D will reduce the value of the reaction at roller E to zero?

    (A) 1221 to the right
    (B) 1461 to the right
    (C) 2344 to the right
    (D) 2441 to the right
    (E) Force must be applied vertically, not horizontally.

Mathematics

Problems 21–22 are based on the following statement.

The position of an object as a function of time is described by

$$x = 4t^3 + 2t^2 - t + 3$$

21. What is the acceleration of the object at $t = 2$?

    (A) 26 (B) 27.5 (C) 41
    (D) 52 (E) 55

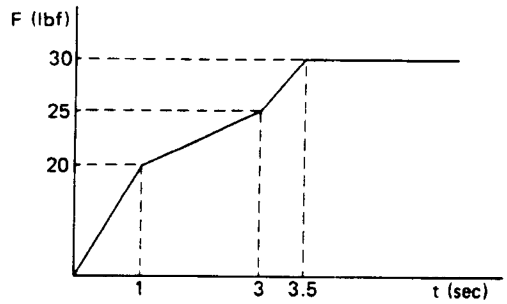
22. What is the total distance traveled from $t = -2$ to $t = 2$?

    (A) −67 (B) $-\frac{421}{27}$ (C) 61
    (D) 67 (E) 118

71

CONTINUE ≫≫

Problems 23–25 refer to the following diagram.



23. What is the velocity of a 4-lbm particle, initially at rest, after 3.5 seconds?

(A) 16.9 ft/sec

(B) 17.2 ft/sec

(C) 69.0 ft/sec

(D) 543 ft/sec

(E) 553 ft/sec

24. What is the change of momentum of the object after 3 seconds if the object has a mass of 40 lbm?

(A) 22.0 lbf-sec

(B) 55 lbf-sec

(C) 67.5 lbf-sec

(D) 68.8 lbf-sec

(E) 550 lbf-sec

25. What is the total linear impulse on a 40-lbm mass after $t = 3.5$ seconds?

(A) 8.5 lbf-sec

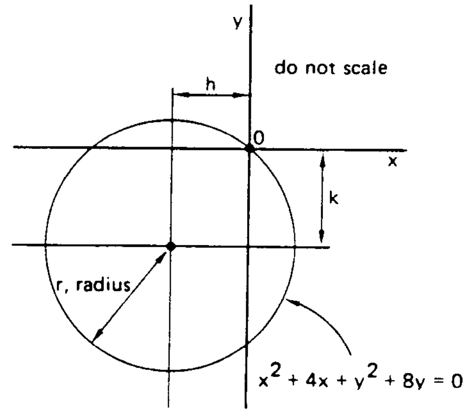
(B) 17.2 lbf-sec

(C) 34.4 lbf-sec

(D) 68.8 lbf-sec

(E) 275.2 lbf-sec

Problems 26–28 refer to the following diagram.



26. What are the coordinates of the circle's center?

(A) $(-4, -8)$ (B) $(2, -4)$ (C) $(-4, -2)$
(D) $(-2, -4)$ (E) $(4, 8)$

27. What is the radius of the circle?

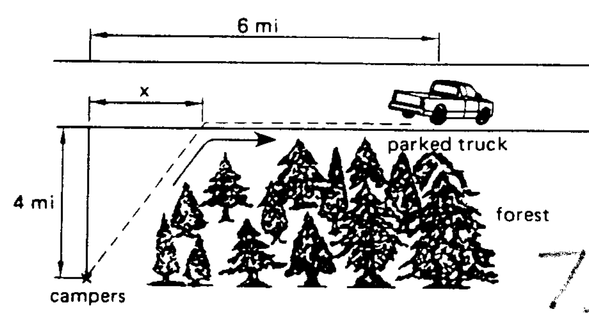
(A) $\sqrt{8}$ (B) $2\sqrt{5}$ (C) $4\sqrt{5}$
(D) 10 (E) 20

28. What is the slope of the line that is tangent to the circle and passes through the origin?

(A) -2 (B) $-\frac{3}{4}$ (C) $-\frac{1}{2}$
(D) $\frac{1}{2}$ (E) 2

Problems 29–33 are based on the following problem statement.

Some campers are walking back to their truck. They are 4 miles from the road on which the truck is parked, and the truck is 6 miles down the road. They walk in a straight line, but not directly back to the road.



72

The campers walk at a rate of 2 miles per hour in the forest and 4 miles per hour on the road.

CONTINUE ≫≫≫

29. What is the equation that describes the total time taken by the campers to walk to the truck?

(A) $t_{\text{total}} = (x^2 + 16) + (6 - x)$

(B) $t_{\text{total}} = \dfrac{\sqrt{x^2 + 16}}{2} + \dfrac{6 - x}{4}$

(C) $t_{\text{total}} = 2\sqrt{x^2 + 16} + (4)(6 - x)$

(D) $t_{\text{total}} = \dfrac{x^2}{4} + \dfrac{x}{2}$

(E) $t_{\text{total}} = \dfrac{4}{6}x^2$

30. What distance must x be to minimize the campers' walking time?

(A) 2 mi (B) $\dfrac{4}{\sqrt{3}}$ mi (C) 4 mi
(D) $4\sqrt{2}$ mi (E) $\dfrac{16}{\sqrt{2}}$ mi

31. What distance must x be to maximize the walking time?

(A) 0 (B) $\sqrt{2}$ mi (C) 3.5 mi
(D) 4 mi (E) 6 mi

32. What is the minimum possible walking time?

(A) 3.23 hr (B) 3.27 hr (C) 3.33 hr
(D) 3.50 hr (E) 3.61 hr

33. What is the maximum possible walking time?

(A) 3.25 hr (B) 3.39 hr (C) 3.45 hr
(D) 3.50 hr (E) 3.61 hr

Problems 34–37 refer to the following equation.

$$2y' = 3xy + x$$

34. The equation is a

(A) first-order polynomial.

(B) second-order polynomial of two variables.

(C) first-order, homogeneous equation.

(D) linear first-order differential equation.

(E) second-order differential equation.

35. How many initial conditions are required to solve the equation for a unique solution?

(A) none

(B) one

(C) two

(D) three

(E) four

36. What is the integrating factor?

(A) $3x$ (B) $\frac{3}{2}x$ (C) $e^{-\frac{3}{2}x^2}$
(D) $e^{-\frac{3}{4}x^2}$ (E) $e^{\frac{3}{2}x^2}$

37. What is the solution?

(A) $y = \ln\left(\frac{3}{2}x^2\right) + C$

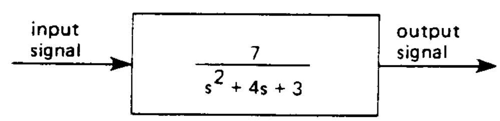
(B) $y = \frac{3}{2}x + C$

(C) $y = Ce^{-\frac{3}{2}x^2} + \dfrac{1}{3}$

(D) $y = Ce^{\frac{3}{4}x^2} - \dfrac{1}{3}$

(E) $y = \dfrac{1}{3}x^2$

Problems 38–40 refer to the following system.



38. What is the output signal as a function of time if a unit impulse function is the input signal?

(A) $\left(\dfrac{7}{2}\right)\left(e^{t} + e^{3t}\right)$

(B) $\left(\dfrac{7}{2}\right)\left(e^{t} + e^{-3t}\right)$

(C) $\left(\dfrac{7}{2}\right)\left(e^{-t} + e^{-3t}\right)$

(D) $\left(\dfrac{7}{2}\right)\left(e^{-3t} - e^{-t}\right)$

(E) $\left(\dfrac{7}{2}\right)\left(e^{-t} - e^{-3t}\right)$

73

CONTINUE >>>

Mathematics

39. What is the steady-state output if the unit step function of height 5 at $t = 0$ is the input signal?

(A) 0 (B) $\frac{3}{35}$ (C) $\frac{35}{6}$
(D) $\frac{35}{3}$ (E) $\frac{7}{3}$

40. What is the steady-state response if the sinusoid $7\sin\left(2t + \frac{\pi}{3}\right)$ is the input signal?

(A) $6.08\angle{-157^\circ}$

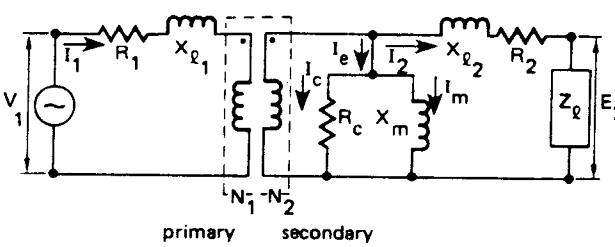
(B) $6.08\angle{-37^\circ}$

(C) $6.08\angle{52.9^\circ}$

(D) $6.08\angle{142.9^\circ}$

(E) $52.9\angle{6.08^\circ}$

Problems 41–50 refer to the figure below.



single-phase transformer
90 kVA
10,000 V primary: 2000 V secondary
60 Hz
power factor: 0.8 leading

$R_1 = 6\ \Omega$ $R_2' = 7\ \Omega$
$X_{L_1} = 31\ \Omega$ $X_{L_2}' = 31\ \Omega$
$X_m' = 55\ \text{k}\Omega$ $R_c' = 120\ \text{k}\Omega$
$|Z_l| = 55\ \Omega$

Note: Primed quantities are equivalent circuit parameters used for Problems 45–50.

41. What is $\mathbf{I}_2$ referred to the secondary side if the secondary voltage is 2000 V?

(A) $36.36\angle{-36.87^\circ}$ A

(B) $36.87\angle{-36.36^\circ}$ A

(C) $33.10\angle{36.87^\circ}$ A

(D) $36.36\angle{36.87^\circ}$ A

(E) $45.82\angle{36.36^\circ}$ A

42. What is $\mathbf{E}_2$ referred to the secondary side?

(A) $2000\angle{0^\circ}$ V

(B) $5000\angle{0^\circ}$ V

(C) $10{,}000\angle{0^\circ}$ V

(D) $2000\angle{90^\circ}$ V

(E) $10{,}000\angle{90^\circ}$ V

43. What is $\mathbf{I}_2$ referred to the primary side?

(A) $181.80\angle{-36.87^\circ}$ A

(B) $36.87\angle{-36.36^\circ}$ A

(C) $7.27\angle{-36.87^\circ}$ A

(D) $7.27\angle{36.87^\circ}$ A

(E) $181.80\angle{36.87^\circ}$ A

44. What is $\mathbf{E}_2$ referred to the primary side?

(A) $400\angle{0^\circ}$ V

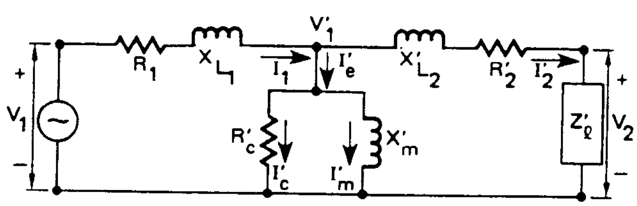
(B) $800\angle{0^\circ}$ V

(C) $5000\angle{0^\circ}$ V

(D) $10{,}000\angle{0^\circ}$ V

(E) $50{,}000\angle{0^\circ}$ V

Problems 45–50 refer to the primary side of the transformer and the following equivalent circuit.



45. What is the current through the inductor, X_m'?

(A) $82.6 \times 10^{-3}\angle{-1.22^\circ}$ A

(B) $82.6 \times 10^{-3}\angle{1.22^\circ}$ A

(C) $88.8 \times 10^{-3}\angle{-179.9^\circ}$ A

(D) $180 \times 10^{-3}\angle{-88.8^\circ}$ A

(E) $180 \times 10^{-3}\angle{88.8^\circ}$ A

74

CONTINUE ≫≫≫

21. -22 $\quad x = 4t^3 + 2t^2 - t + 3$

21. What is acceleration when $t = 2$?

$$x = 4t^3 + 2t^2 - t + 3$$

$$\dot{x} = 12t^2 + 4t - 1$$

$$\ddot{x} = 24t + 4$$

$$\ddot{x}(t=2) = 24 \cdot 2 + 4 = 48 + 4 = 52$$

$\boxed{\text{Answer D}}$

22. What is the total distance traveled from $t = -2$ to $t = 2$?

Does it reverse motion? If so then $v = 0$ in $-2 < t < 2$.

$$\dot{x} = 12t^2 + 4t - 1 = (6t-1)(2t+1) = 0 \quad t = -\tfrac{1}{2}, +\tfrac{1}{6}$$

$$x(-2) = -32 + 8 + 2 + 3 = -19$$

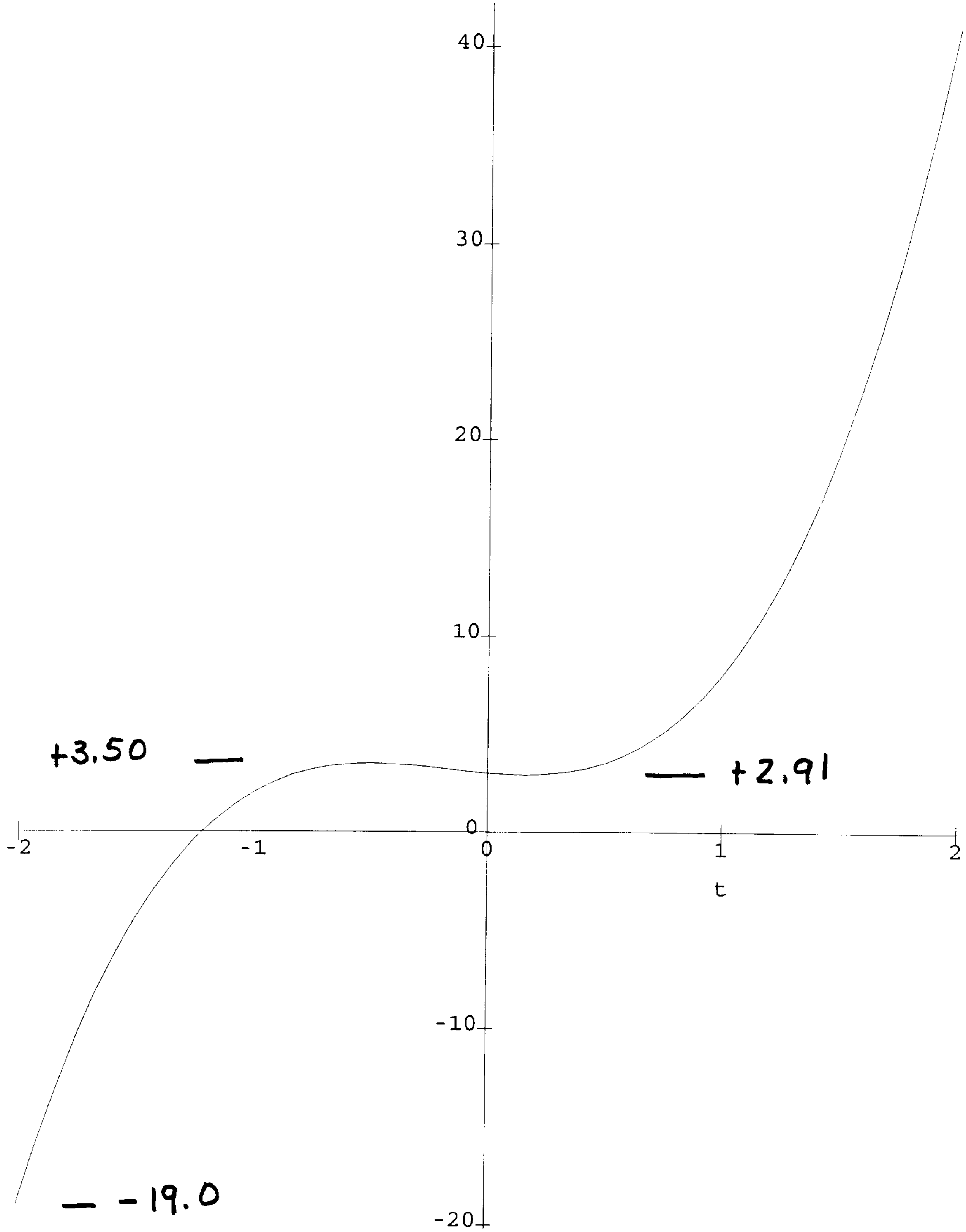
$$x(-\tfrac{1}{2}) = -\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + 3 = 3.5$$

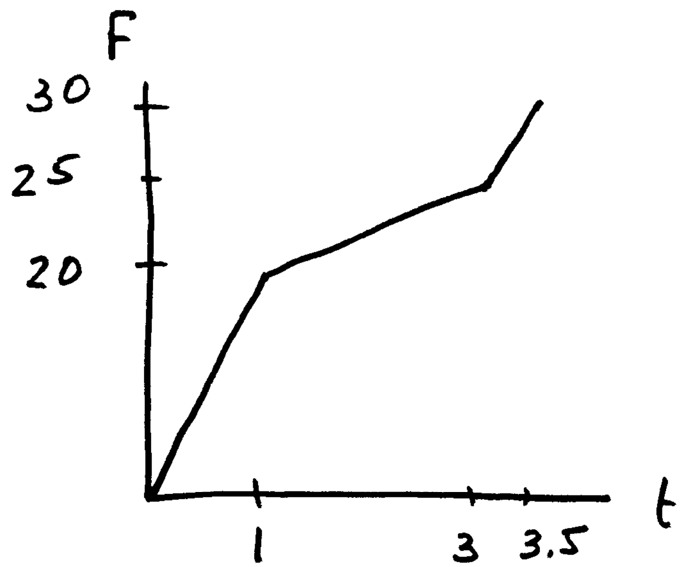
$$x(+\tfrac{1}{6}) = \frac{4}{216} + \frac{2}{36} - \frac{1}{6} + 3 = \frac{157}{54} = 2.907$$

$$x(+2) = 32 + 8 - 2 + 3 = 41$$

$$S = (3.5 - (-19)) + |2.907 - 3.5| + (41 - 2.907) = 22.5 + 0.593 + 38.09$$

x=4t^3 + 2t^2 - t +3, -2 < t < 2.





$m = 4 lbm$

23. Impulse = Change of momentum

$m = 4\ lbm \quad V_o = 0, \quad V_f = ?$

$$I = \left(\frac{0+20}{2}\right)(1) + \left(\frac{20+25}{2}\right)(2) + \left(\frac{25+30}{2}\right)\left(\frac{1}{2}\right)$$

$$= 10 + 45 + \frac{55}{4} = 68.75 \text{ lb-sec.}$$

$$\Delta mv = m(V_f - V_o) = \frac{4}{32.2}(V_f - 0)$$

$$\frac{4 V_f}{32.2} = 68.75 \qquad V_f = 553.44 \text{ ft/sec}$$

$$\boxed{Ans = 553 \text{ ft/sec} = E}$$

24. Change of momentum after 3 sec if M = 40 l

$$\text{Impulse} = 10 + 45 = 55 \text{ lb}_f \text{ sec}$$

$$\boxed{Ans = 55 \text{ lb}_f\text{-sec} = B}$$

25. What is the total linear impulse on a 40-lbm mass after 3.5 sec?

$$\text{Impulse} = 10 + 45 + \frac{55}{4} = 68.75\ \text{lb}_f\text{-sec}$$

$$\boxed{\text{Ans} = 68.8\ \text{lb}_f\text{-sec} = D}$$

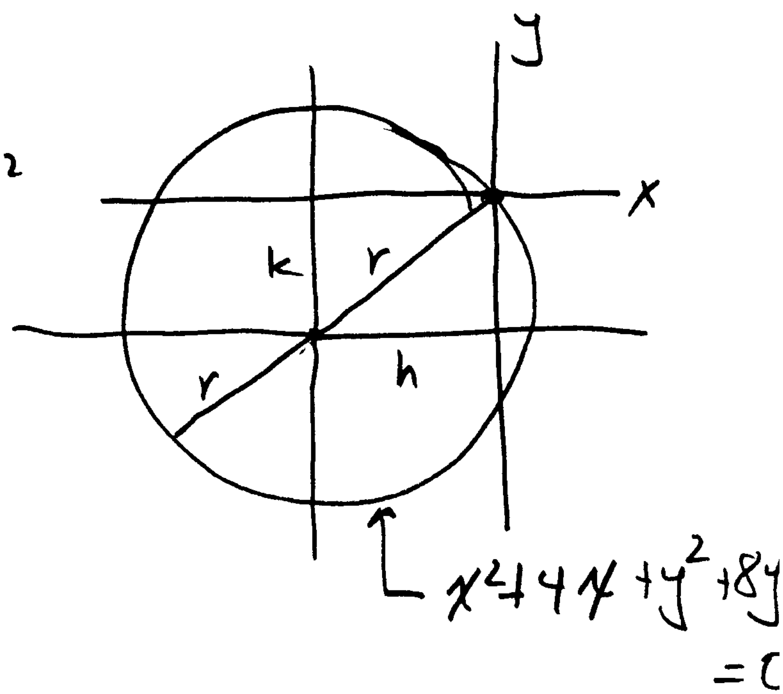
26. $(x-h)^2 + (y-k)^2 = r^2$

$x^2 - 2xh + h^2 + y^2 - 2ky + k^2 = r^2$

$-2hx = +4x;\ h = -2$

$-2ky = 8y;\ k = -4$

$$\boxed{\text{Ans} = (-2,-4) = D}$$



27. What is r?

$r^2 = h^2 + k^2 = (-2)^2 + (-4)^2 = 4 + 16 = 20$

$r = \sqrt{4 \cdot 5} = \boxed{2\sqrt{5} = \text{Answer B}}$

28. What is the slope of the line tangent to the circle and passes through the origin.

$$x^2 + 4x + y^2 + 8y = 0$$

Note: This circle does pass through the origi

Differentiate wrt x

$$2x + 4 + 2y\frac{dy}{dx} + 8\frac{dy}{dx} = 0$$

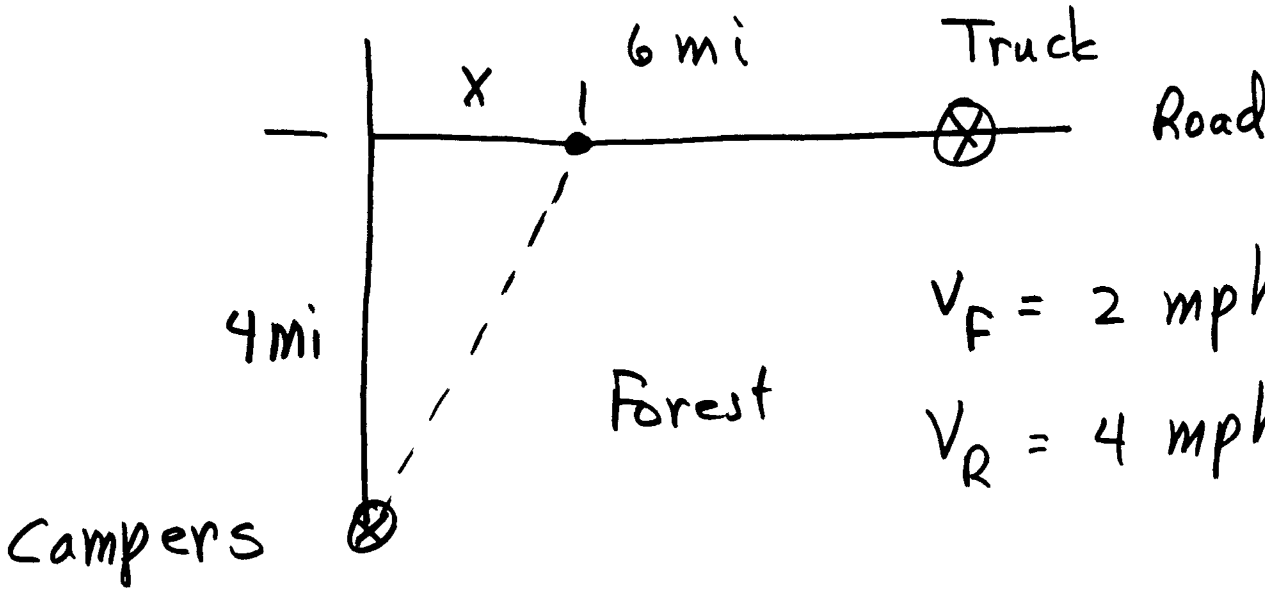
Check value of y' when $x = y = 0$

$$2 \cdot 0 + 4 + 2 \cdot 0 \cdot y' + 8y' = 0$$

$$4 + 8y' = 0$$

$$\boxed{y' = -1/2 = \text{Ans C}}$$

29-33. Campers walk back to truck



$V_F = 2$ mph

$V_R = 4$ mph

29. What is equation for total time to walk to truc

$vt = d \quad t = d/v$

$$t = t_F + t_R$$

$d_f = \sqrt{4^2 + x^2} \quad t_f = \frac{d_f}{2}$

$d_R = 6 - x \quad t_R = \frac{d_R}{2}$

$$\boxed{t = \frac{\sqrt{16 + x^2}}{2} + \frac{6 - x}{4} = \text{Ans B}}$$

30. What x minimizes time?

$$\frac{dt}{dx} = \frac{1}{2} \cdot \frac{2x}{2\sqrt{16 + x^2}} - \frac{1}{4} = \frac{x}{2\sqrt{16 + x^2}} - \frac{1}{4} = 0$$

$2x = \sqrt{16 + x^2}$

$4x^2 = 16 + x^2$

$3x^2 = 16$

$$x = \sqrt{16/3} = \boxed{4/\sqrt{3} \text{ mil} \quad \text{Ans B}}$$

31. What value of x maximizes the time?

Case 1 walk straigh to road, $x=0$

$$t = \frac{\sqrt{16}}{2} + \frac{6}{4} = \frac{4}{2} + \frac{6}{4} = 2 + 1.5 = 3.5\text{ hr}$$

Case 2 walk straight to truck, $x=6$

$$t = \frac{\sqrt{16+36}}{2} + \frac{6-6}{0} = \frac{\sqrt{52}}{2} = \frac{2\sqrt{13}}{2} = \overset{3.61}{\cancel{7.21}}\text{ hr}$$

$$\boxed{\text{Ans } x = 6 \text{ miles} \quad E}$$

32. What is the minimum walking time?

$$t = \frac{\sqrt{16 + \frac{16}{3}}}{2} + \frac{6 - 4/\sqrt{3}}{4} = \frac{4\sqrt{4/3}}{2} + \frac{6}{4} - \frac{1}{\sqrt{3}}$$

$$= \frac{4}{\sqrt{3}} + 1.5 - \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} + 1.5 = \sqrt{3} + 1.5 = 3.23\text{ hr}$$

$$\boxed{t = 3.23\text{ hr} = \text{Ans A}}$$

33. What is the maximum walking time?

$$\boxed{t = 3.61\text{ hr} = \text{Ans E}}$$

Probs 34-37

$$2y' = 3xy + x.$$

34. The equation is - A - First order polynomial
B - 2nd Order polynomial in 2 variables
C - 1st Order homogeneous equa
→ D - Linear 1st Order DE
E - 2nd Order DE

35. How many Initial Conditions are Required?

$\boxed{\text{Ans B - One}}$

36. What is the I F ?

$$2y' - 3xy = x$$

$$y' - \frac{3}{2}x\, y = \frac{1}{2}x$$

$$P(x) = -\frac{3}{2}x \qquad \int P(x)\,dx = -\frac{3}{4}x^2$$

$$IF = e^{-\frac{3}{4}x^2} = \boxed{\text{Ans D}}$$

37. What is Solution

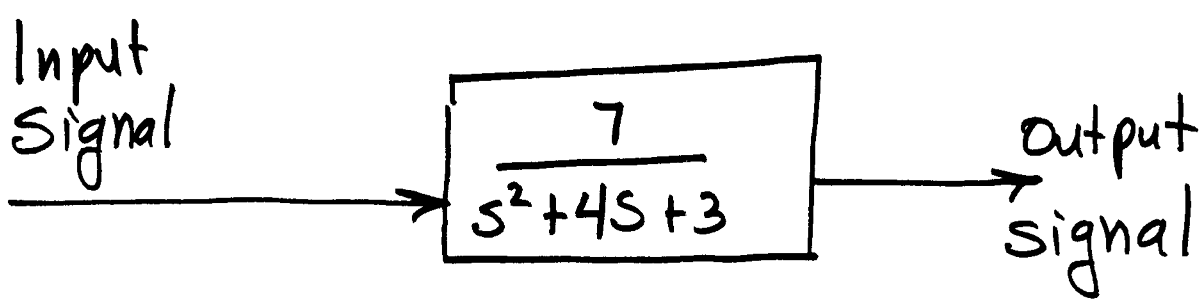
$$y e^{\int P(x)dx} = \int Q e^{\int P(x)dx} dx + C$$

$$y e^{-\frac{3}{4}x^2} = \int \left(\frac{1}{2}x\right) e^{-\frac{3}{4}x^2} dx + C$$

$$= -\frac{1}{3} \int e^{-\frac{3}{4}x^2} \left(-\frac{3}{2}x\,dx\right) + C$$

$$y e^{-\frac{3}{4}x^2} = -\frac{1}{3} e^{-\frac{3}{4}x^2} + C$$

$$\boxed{y = -\frac{1}{3} + C e^{+\frac{3}{4}x^2} = \text{Ans D}}$$



Output signal transform = $R(s)$

Input signal transform = $F(s)$

Then

$$\frac{\text{Output Transform}}{\text{Input transform}} = \text{Black Box Transform} = \frac{7}{s^2+4s+}$$

38. Output Signal if the input is the unit impu

Input = $\delta(t)$ $F(s) = 1$

$$R(s) = \frac{7}{s^2+4s+3} = \frac{7}{(s+3)(s+1)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$7 = A(s+3) + B(s+1)$$

$s=-1$, $7 = A(-1+3) = 2A$; $A = 7/2$

$s=-3$, $7 = 0 + B(-2)$; $B = -7/2$

$$R(s) = \frac{7/2}{s+1} - \frac{7/2}{s+3}$$

$$\boxed{\text{Output} = \frac{7}{2}\left(e^{-t} - e^{-3t}\right) = \text{Ans E}}$$

39. Input = $5u(t)$; $u(t)$ = unit step function.

What is steady state response?

$F(t) = 5u(t)$; $F(s) = \frac{5}{s}$

$$R(s) = \frac{5}{s} \cdot \frac{7}{s^2+4s+3} = \frac{35}{s(s+1)(s+3)}$$

$$R(t)\Big|_{t\to\infty} = s \cdot R(s)\Big|_{s\to 0}$$

$$= s \cdot \frac{35}{s(s+1)(s+3)}\Big|_{s\to 0}$$

$$= \frac{35}{(s+1)(s+3)}\Big|_{s\to 0} = \frac{35}{3}$$

$\boxed{\text{Ans D}}$

<u>OR</u>

$$\frac{35}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$35 = A(s+1)(s+3) + Bs(s+3) + Cs(s+1)$$

Let $s=0$; $35 = A(1)(3)$; $A = 35/3$.

$R(t) = \frac{35}{3} + Be^{-t} + Ce^{-3t}$; $\boxed{R(t\to\infty) = \frac{35}{3}}$

40. What is steady state response if Input is $7\sin(2t + \pi/3)$?

Angle $= \pi/3$ rad $= 60°$

$\omega = 2$ rad/sec.

For $s = j2$ $\quad (j^2 = -1)$

$$\frac{7}{s^2+4s+3} = \frac{7}{(j2)^2 + 4(j2) + 3} = \frac{7}{-1 + j8}$$

$$= \frac{7}{8.062 \angle 97.13} = 0.868 \angle -97.13$$

Input $= 7 \angle +60$

Output $= 7 \angle +60 \cdot 0.868 \angle -97.13$

$$= \boxed{6.08 \angle -37.13° = \text{Ans B}}$$