

ECONOMICS

FUNDAMENTALS OF
ENGINEERING REVIEW

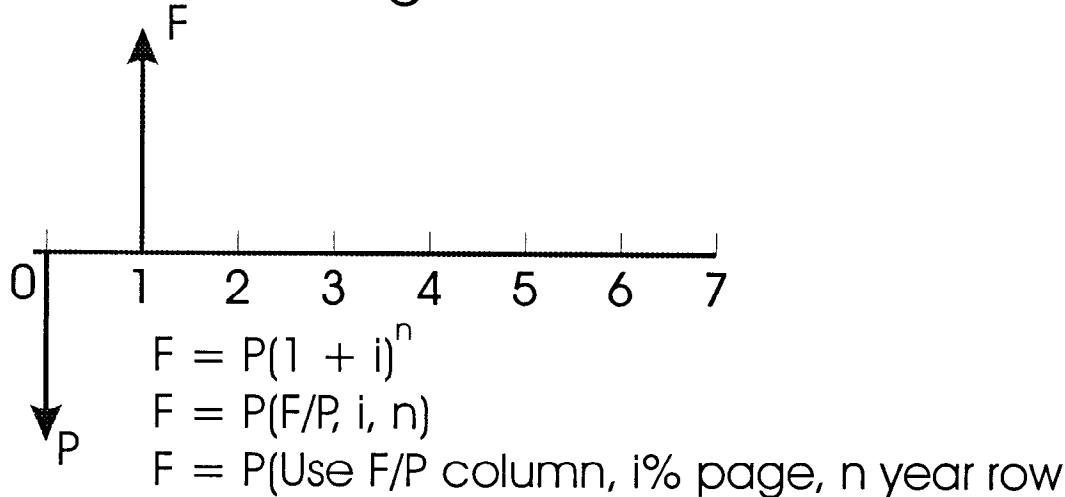
Lee L. Lowery, Jr., PhD, P.E.

Presented to Fort Hood
4/15/98

Moving the value of money from the present to the future:

Say we borrow \$20,000 from the bank for a period of one year at an annual interest rate of 6%. How much do we owe the bank after one year?

Find Future given Present

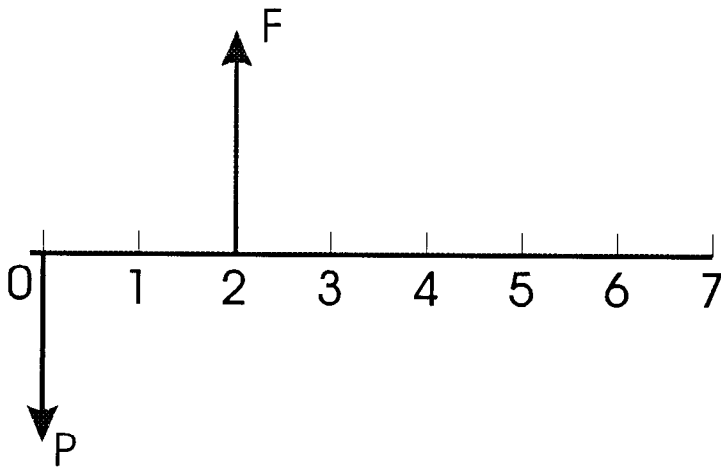


$$\begin{aligned} F &= \$20,000 + 0.06(\$20,000) \\ &= \$20,000 (1 + 0.06) \\ &= \$20,000 (1 + 0.06)^1 \\ &= \$20,000 (1.06) \\ &= \$21,200 \end{aligned}$$

or, in economics shorthand:

$$\begin{aligned} F &= P (F/P, i, n) = P(\text{a factor from a table}) = \\ &= P(\text{Use F/P column, } i\% \text{ rate, } n \text{ years or periods}) \\ &= \$20,000 (1.06) \\ &= \$21,200 \end{aligned}$$

Please note that we will try to use 6% interest for all of the problems, since that rate is listed in the reference manual.



Now say we borrow the money for two years:

$$\begin{aligned}
 F &= \$20,000 + 0.06(\$20,000) + 0.06(\$20,000) + 0.06(0.06 * \$20,000) \\
 &= \$20,000 (1 + 0.06)(1 + 0.06) \\
 &= \$20,000 (1 + 0.06)^2 \\
 &= \$20,000 (1.1236) \\
 &= \$22,472
 \end{aligned}$$

or, in economics shorthand:

$$\begin{aligned}
 F &= P (F/P, i, n) = P(\text{a factor from a table}) = \\
 &= P(\text{Use F/P column, } i\% \text{ rate, } n \text{ years or periods}) \\
 &= \$20,000 (1.1236) = \$22,472
 \end{aligned}$$

Notice how you know what factor you need to use from the tables by the fact that the "P's" appear to cancel out of the factor, leaving you with F, which is what you wanted. Thus, if you have money at M and you want to change it to money at a different time (or type) Q, then you would write:

$$Q = M(Q/M, i, n)$$

Notice how the M's appear to cancel, leaving what you want - Q.

Similarly, if I want to take a given Annuity or Annual amount and transfer it into a Present amount, I would write:

What I want = What I got times a factor from the tables:

$$P = A(P/A, i, n)$$

To transfer a given Future value into an Annuity or Annual amount I would write

$$A = F(A/F, i, n)$$

To GET the Gradient FROM an Annual amount I would write:

$$G = A(G/A, i, n)$$

To GET the Present value OF a (given) Future value, I write:

$$P = F(P/F, i, n)$$

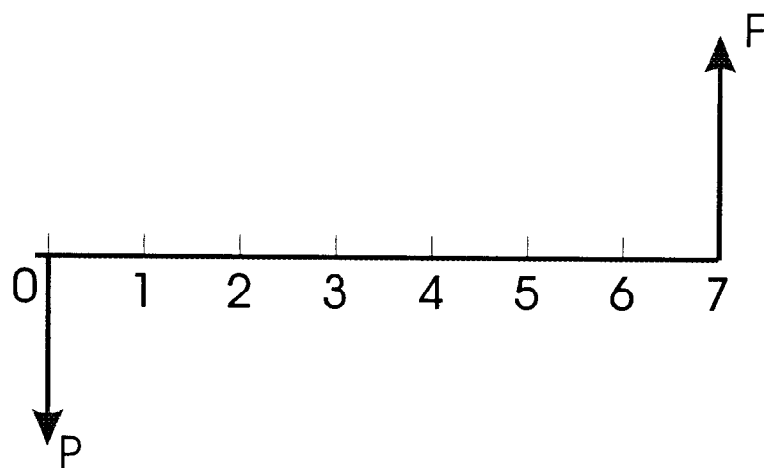
Note also that sometimes the reference manual does not give a column, such as for G/A. In that case:

$$\begin{aligned} G &= A(G/A, i, n) \\ &= A / (A/G, i, n) \end{aligned}$$

Also, you may have to piece together what you want:

$$\begin{aligned} F &= G(F/G, i, n) \quad (\text{but } F/G \text{ is not in the reference manual}) \\ &= G(F/A, i, n)(A/G, i, n) \quad (\text{which are both in the manual.}) \end{aligned}$$

Now what if we borrow the money for 7 years:



$$F = \$20,000*$$

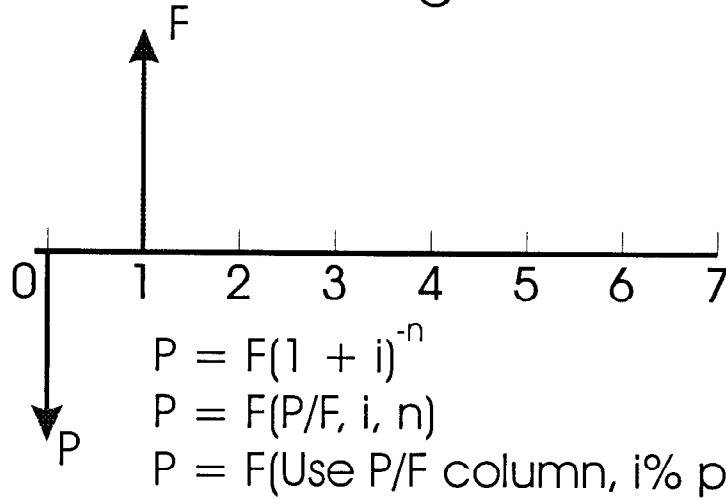
$$\begin{aligned} & (1+0.06)(1+0.06)(1+0.06)(1+0.06)(1+0.06)(1+0.06)(1+0.06) \\ &= \$20,000 (1 + 0.06)^7 \\ &= \$20,000 (1.5036) \\ &= \$30,072 \end{aligned}$$

or, in economics shorthand:

$$\begin{aligned} F &= P (F/P, i, n) = P(\text{a factor from a table}) = \\ &= P(\text{Use F/P column, } i\% \text{ rate, } n \text{ years or periods}) \\ &= \$20,000 (1.5036) = \$30,072 \end{aligned}$$

Moving the value of money from the future to the present:

Find Present given Future



Now assume we would like to deposit a sum of money in the bank and have it earn interest at 6% for one year, and that we desire the bank to return \$20,000 to us at the end of that year. How much must we invest?

Since $F = P(1+i)^n$ then $P = F(1+i)^{-n}$

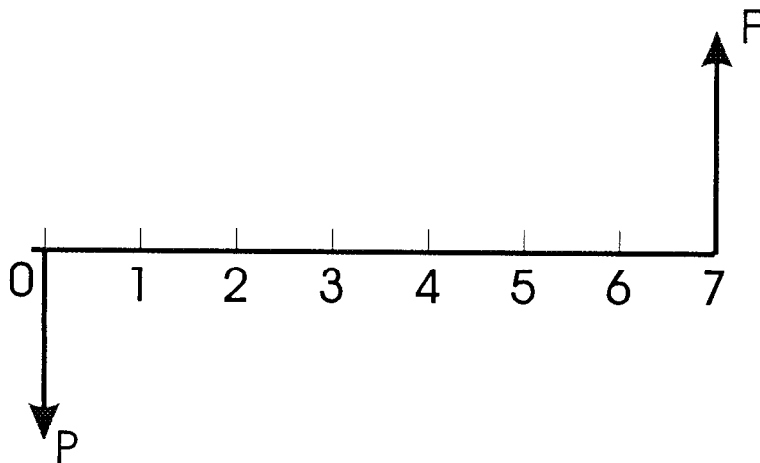
So the P you would need to deposit now to get back \$20,000 at the end of one year at 6% interest would be:

$$P = \$20,000(1+0.06)^{-1} = \$18,868$$

or in ecospeak:

$P = F(P/F, i, n)$ = Multiply the future value you want times a factor designed to convert future money back to present money. Use the P/F column on the 6% page, on the 1 year line.

$$P = \$20,000(0.9434) = \$18,868$$

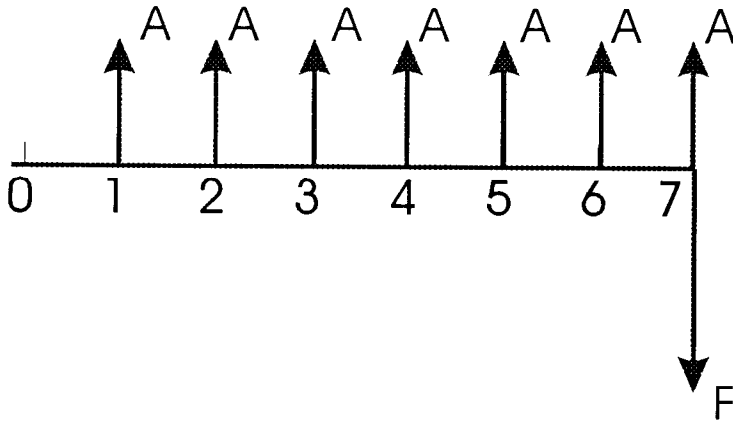


or, for 7 years: $P = \$20,000(0.6651) = \$13,302$

Finding the Annual contribution required to get some desired Future value:

How much money would we have to invest annually during the next 7 years to be able to withdraw \$1678.76 at the end of that 7 year period? Now at this time some discussion arises - namely, should we assume that in answering that question the person intends to deposit the first installment now? Today? Immediately? Or should we assume that having today decided to do this, it will take him a year to get the first payment together? Either way is fine, but we don't want to have a set of interest tables which say "7 year investment, first payment made today" and a second set of tables which say "7 year investment, first payment made one year from today." So we will pick one or the other, and we have all agreed to pick the latter. Namely, that having decided today to start investing money, we will have the first amount ready for investment at the end of the first year. Thus the cash flow diagram will look:

Find A given F



$$A = F(i / (1+i)^n - 1)$$

$$A = F(A/F, i, n)$$

$$A = F(\text{Use } A/F \text{ column, } i\% \text{ page, } n \text{ year row})$$

Thus in our case, $A = F(A/F, i, n)$ where $(A/F, i, n)$ can be found from the tables in the Reference Manual as 0.1191,

$$\text{Thus } A = F(A/F, i, n)$$

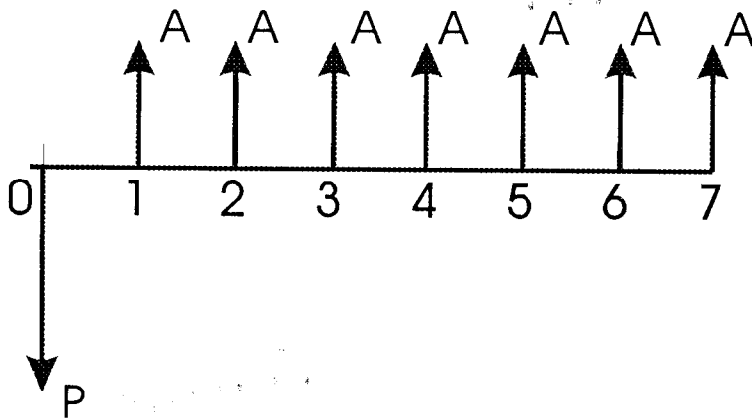
$$A = F(\text{Use } A/F \text{ column, } 6\% \text{ page, } 7 \text{ year row})$$

$$A = \$1678.76 (0.1191) = \$199.94$$

Finding the Annual withdrawal permitted given a Present contribution:

Now what if we are given a gift of \$1116.47 and wish to invest it in the bank at 6% for 7 years, taking out an equal amount each year? How much would the bank let us withdraw each year?

Find A given P



$$A = P(i(1+i)^n / ((1+i)^n - 1))$$

$$A = P(A/P, i, n)$$

A = P(Use A/P column, 1% page, n year row)

$$A = P(A/P, i, n)$$

$$= \$1116.47(\text{Use A/P column, } i\% \text{ page, } n \text{ year row})$$

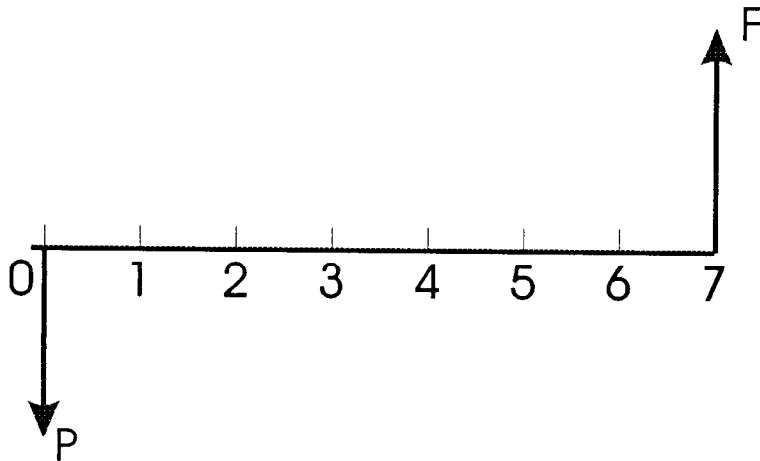
$$= \$1116.47(0.1791)$$

$$= \$ 199.96$$

This is the same amount that we found on the previous problem, (within the decimals of accuracy we are using), and we are in effect saying that \$199.94 invested for 7 years has the same value at the end of that 7 years, as it has at the beginning of that 7 years, all in one lump sum.

Let's check the accuracy of this statement:

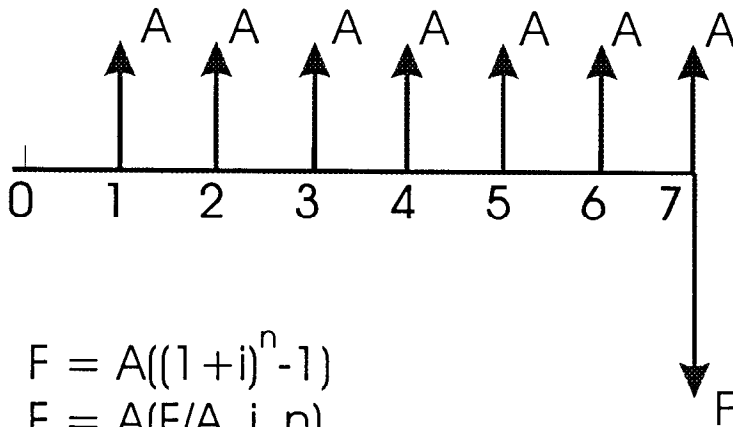
Since an annual investment of \$199.94 for 7 years at 6% has an equivalent Present value of \$1116.47, and an equivalent Future value of \$1678.76, is P equivalent to F?



$$\begin{aligned}
 F &= P(F/P, i, n) \\
 &= \$1116.47(\text{Use } F/P \text{ column, } 6\% \text{ page, } 7 \text{ year row}) \\
 &= \$1116.47(1.5036) \\
 &= \$1678.72 \text{ which is the same as we found earlier } (\$1678.76)
 \end{aligned}$$

Finding F knowing A:

Find F given A



$$\begin{aligned}
 F &= A((1+i)^n - 1) \\
 F &= A(F/A, i, n) \\
 F &= A(\text{Use } F/A \text{ column, } i\% \text{ page, } n \text{ year row})
 \end{aligned}$$

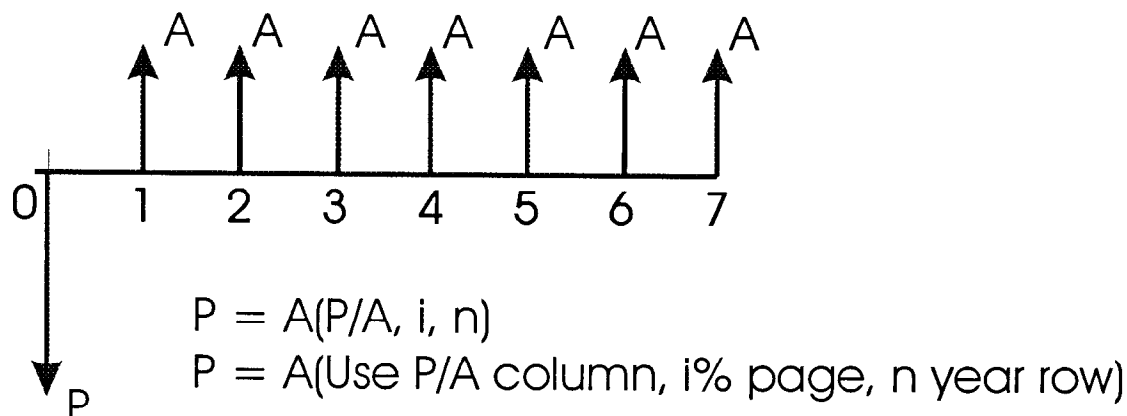
What if we want to invest \$199.96 for 7 years. How much will we be able to collect at the end of 7 years? *1678.76*

$$F = A(F/A, i\%, n)$$

$$\begin{aligned}
 F &= \$199.96(\text{Use } F/A \text{ column, } 6\% \text{ table, } 7 \text{ year row}) \\
 &= \$199.96(8.3938) \\
 &= \$1678.42 - \text{same as before } (\$1678.76)
 \end{aligned}$$

Finding P given A:

Find P given A



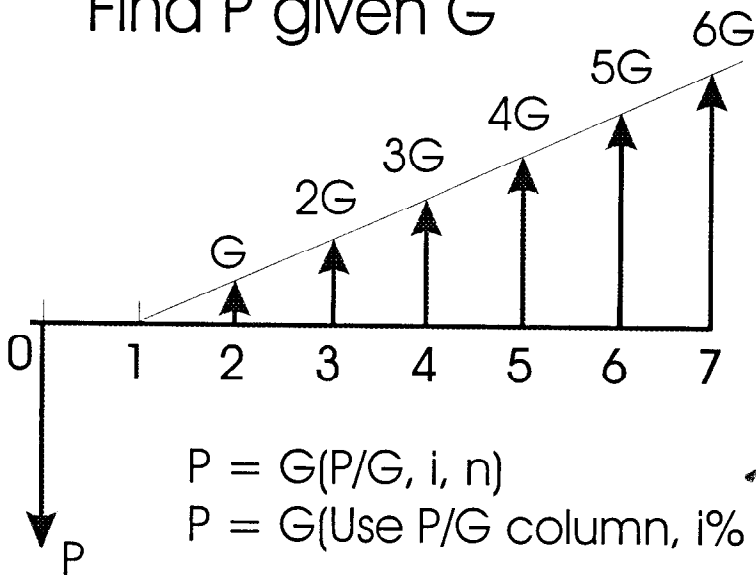
If we wanted to determine how much money to put in a bank today at 6% so that we could withdraw equal amounts of \$199.96 for the next 7 years, starting at the end of the first year:

$$\begin{aligned}
 P &= A(P/A, i, n) \\
 &= \$199.96(\text{Use } P/A \text{ column on } 6\% \text{ table on } 7 \text{ year row}) \\
 &= \$199.96(5.5824) \\
 &= \$1116.15 \text{ (essentially the same as used earlier)}
 \end{aligned}$$

Gradient cash flows:

Now sometimes we have constantly increasing values of money as the years progress, called a "gradient". The cash flow diagram must as before be started from a commonly agreed on year, or no standard tables would be possible. The agreement has been that gradient cash flows will look:

Find P given G



Note also that $P = G(P/G) = G(P/A)(A/G)$

Could we have agreed that the "G" started at year 1? Sure, we could have, but we didn't. Could we have agreed that the "G" started at year zero? Yep. But we didn't. So if you want to use our tables, you will have to position your first non-zero value of "G" as having been deposited, or withdrawn from the bank at the end of the second year. It will become obvious why the "zero G" value was positioned at year 1 when we combine Annual and Gradient cash flows.

Finding Present deposit to yield a given Gradient:

Question: How much money do we need to invest today, at 6%, to be able to withdraw \$0 at the end of year 1, \$100 at the end of year 2, \$200 at the end of year 3, ... , and \$600 at the end of year 7?

$$\begin{aligned}
 P &= G(P/G, i\%, n) \\
 &= \$100(\text{Use } P/G \text{ column, } 6\% \text{ interest, } 7 \text{ year row}) \\
 &= \$100(15.4497) \\
 &= \$1544.97
 \end{aligned}$$

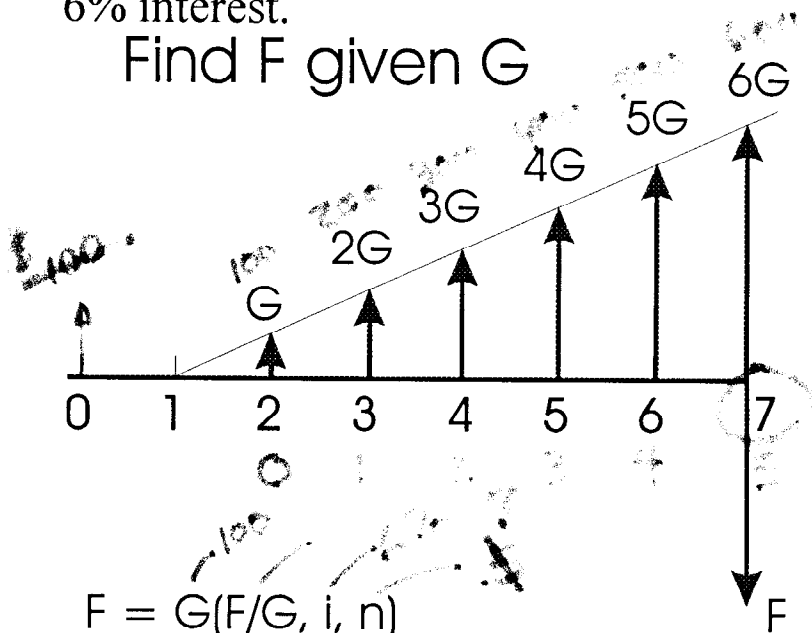
Note also that:

$$\begin{aligned}
 P &= G(P/G, i, n) \\
 &= G(P/A, i, n)(A/G, i, n) \\
 &= \$100(5.5824)(2.7676) \\
 &= \$1544.99 \text{ (Close enough)}
 \end{aligned}$$

Finding F given G:

Question: How much money could we take out of the bank after 5 years if we were willing to invest \$100 today, \$200 at the end of the first year, \$300 at the end of the second year, \$400 at the end of the third year, \$500 at the end of the fourth year, and \$600 at the end of the fifth year? Use 6% interest.

Find F given G



$$F = G(F/G, i, n)$$

$$F = G(\text{Use } F/G \text{ column, } i\% \text{ page, } n \text{ year row})$$

But we don't HAVE an F/G column, so:

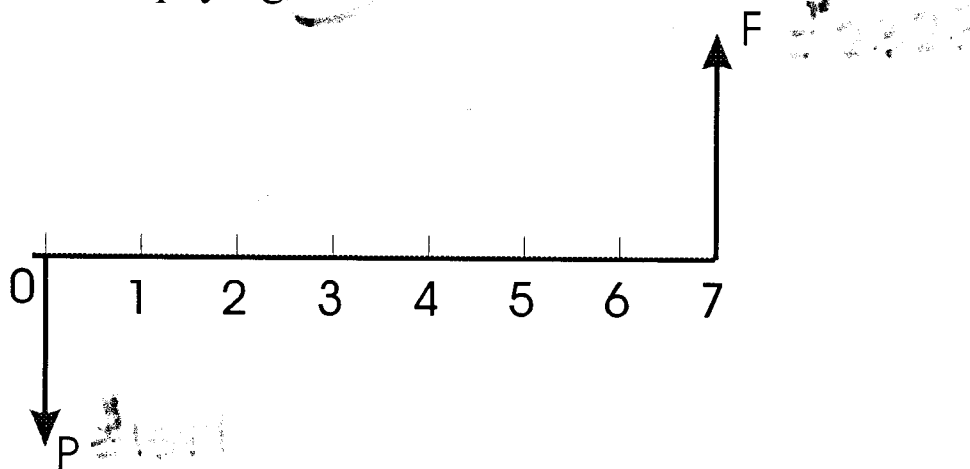
$$F = G(F/A)(A/G)$$

Note that in this case the problem is not stated in the "standard" form which we must use if we are to use the standard interest tables. The

standard tables are set up assuming that today you decide to make the investment. At the end of the first year you invest 0G. At the end of the second year you invest 1G. At the end of the third year you invest 2g, etc. Thus in order to use the standard tables you will have to shift time back as shown in the cash flow diagram shown above. Also note that the reference manual does not give F/G, so you will have to combine F/A and A/G to get F/G:

$$\begin{aligned}
 F &= G(F/A, i, n)(A/G, i, n) \\
 &= \$100(\text{Use F/A column, 6\% table, 7 yrs})(\text{Use A/G table, 6\%, 7 yrs}) \\
 &= \$100(8.3938)(2.7676) \\
 &= \$2323.07
 \end{aligned}$$

Now, let's check the equivalence of the previous two problems - i.e. is \$1544.99 today the same as having \$2323.07 at the end of 7 years, if the bank is paying 6% interest?



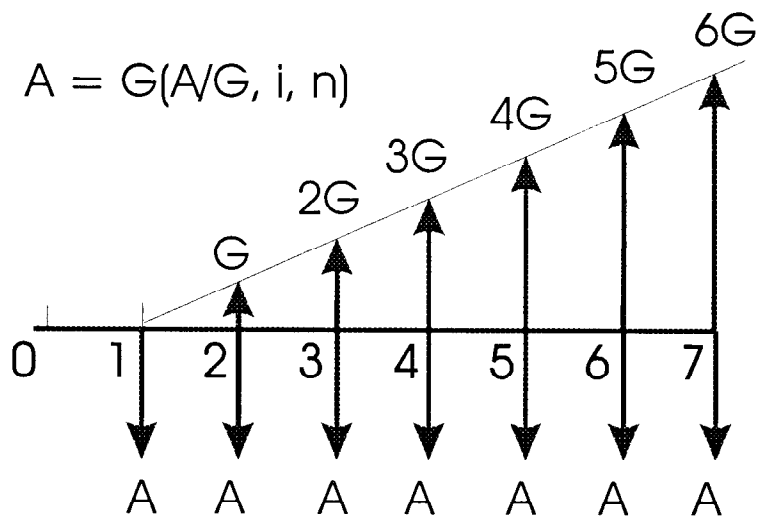
If $P = \$1,544.99$, what is F ?

$$\begin{aligned}
 F &= P(F/P, i, n) \\
 &= \$1544.99(\text{Use F/P column, 6\%, 7 year row}) \\
 &= \$1544.99(1.5036) \\
 &= \$2323.05 \quad \text{OK}
 \end{aligned}$$

Find A given G:

Question: If we were able to invest a Gradient = \$100 at the end of each year for 7 years at 6% interest (i.e. \$0 at the end of year 1, \$100 at the end of year 2, \$200 at the end of year 3, ... , and \$600 at the end of year 7, what Annual amount or Annuity could we take out of the bank starting at the end of the first year and continuing until the end of the seventh year?

Find A given G



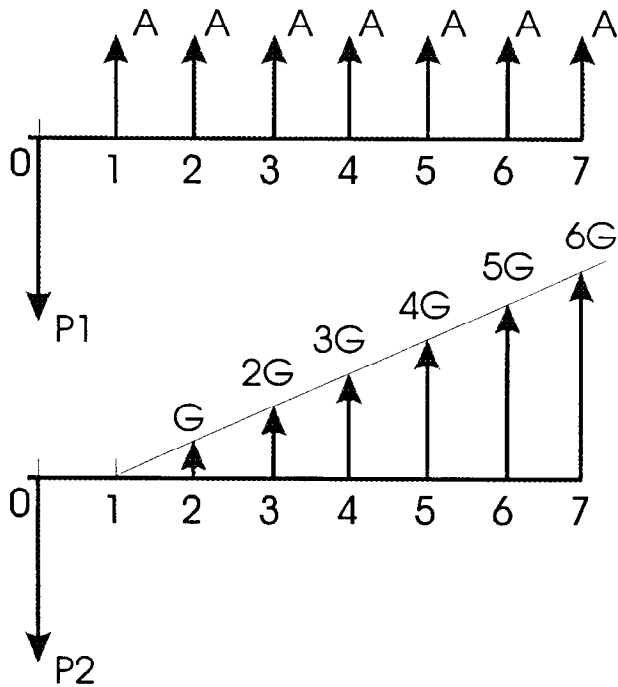
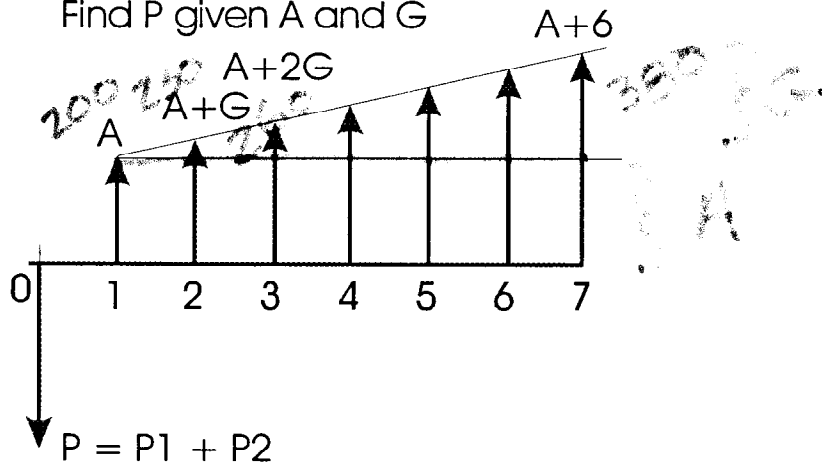
$A = G(A/G, i, n)$
= \$100(Use A/G column, 6% interest, 7 year row)
= \$100(2.7676)
= \$276.76

Find P given A and G:

Question: How much money must you put in the bank TODAY, to be able to draw out \$200 at the end of the first year, \$230 at the end of the

second year, \$260 at the end of the next year, and so on, until taking out \$380 at the end of the seventh year?

Find P given A and G



$$P1 = A(P/A, i, n) = \$200(P/A, 6\%, 7 \text{ yrs}) = \$200(5.5824) = \$1116.48$$

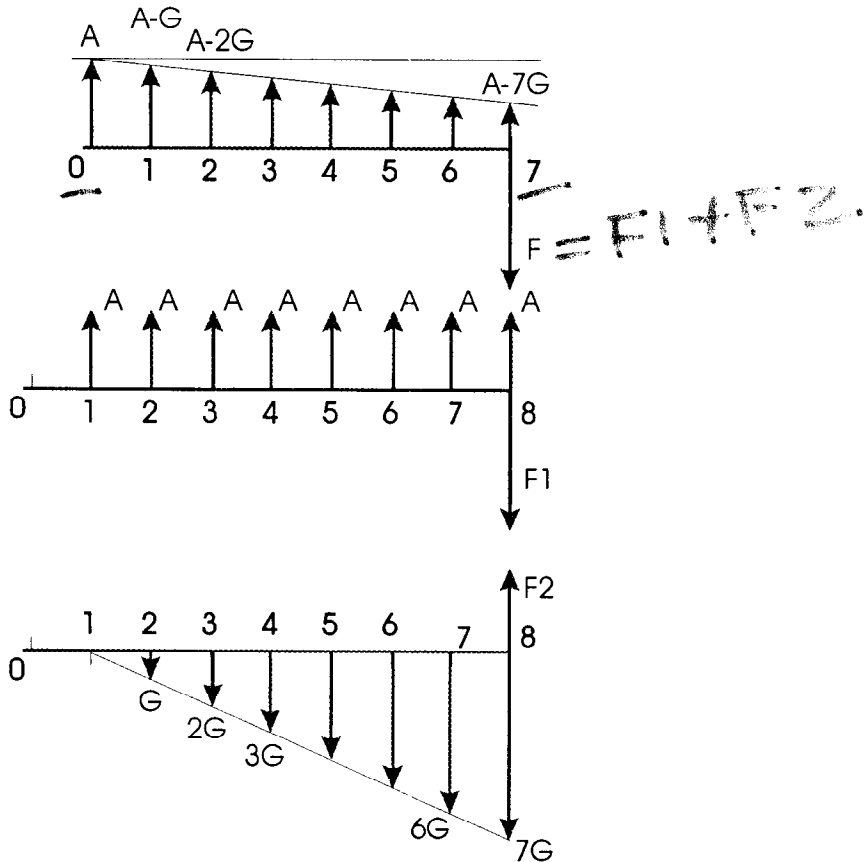
$$P2 = G(P/G, i, n) = \$30(P/G, 6\%, 7 \text{ yrs}) = \$30(15.4497) = \$463.49$$

$$\text{So } P = P1 + P2 = \$1579.97$$

This should demonstrate why the gradient STARTS ON THE 2nd YEAR!

Now what if we were able to invest \$600 today (note - this is not standard!), \$550 at the end of the year 1, \$500 at the end of year 2, and so on until finally investing \$250 at the end of year 7? How much would the money invested at 6% be worth?

Find P given A and -G



$$F1 = A(F/A, 6\%, 8 \text{ years!!!!}) = \$600(9.8975) = \$5938.50$$

$$F2 = G(F/G, 6\%, 8 \text{ years!!!!}) = G(F/A, 6\%, 8 \text{ years})(A/G, 6\%, 8 \text{ years})$$

$$= \$50(9.8975)(3.1952) = \$1581.22 \text{ --- NEG.}$$

$$\text{Making } F = F1 - F2 = \$4357.28$$

Compounding Non-Annually:

Now what if you have a "nominal" interest rate of 6%, but if you will bring your \$6000 to my bank I will compound your interest every quarter

(every 3 months) rather than only yearly. That means I am willing to pay you $6\%/4 = 1.5\%$ interest on your money every 3 months, which means you will make more interest on your money because you will start getting "interest on interest" sooner than at the end of each year. For example, at the end of the first 3 months, I will put another $\$6000 \times 0.015 = \90 in your account, which will then earn interest for the next 9 months. Thus you now have $\$6090$ in your account. Then, after another 3 months, I will add to your account $\$6090 \times 0.015 = \91.35 , for a total of $\$6181.35$. At the end of the next 3 months I will add $\$6181.35 \times 0.015 = \92.72 , for a total of $\$6274.07$, and finally at the end of the next 3 months I will add another $\$6274.07 \times 0.015 = \94.11 for a total of $\$6368.18$. Compare this with the $\$6000(1+0.06) = \6360.00 you would have had had I not compounded the interest quarterly. So you are $\$8.18$ better off.

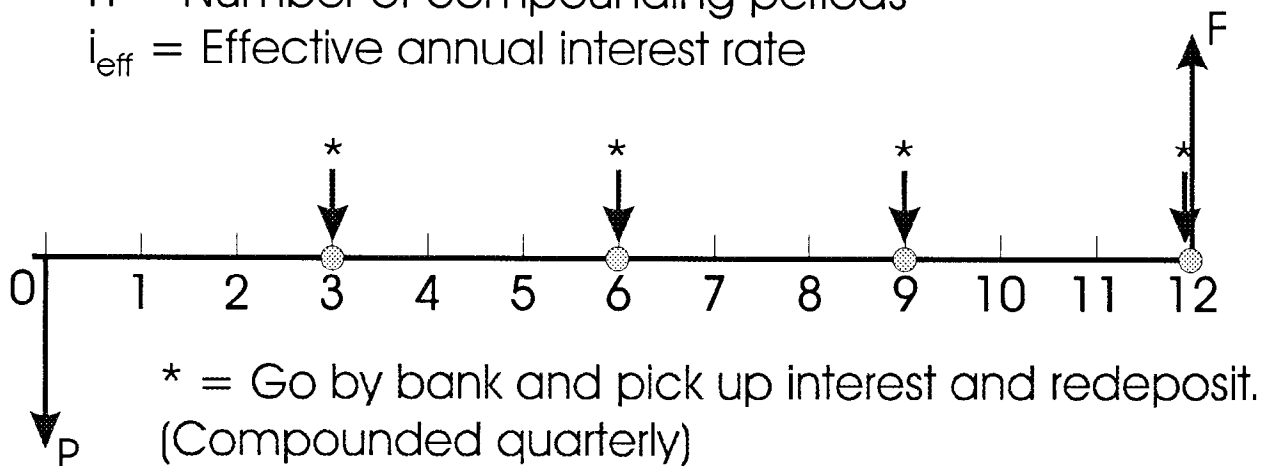
Non-Annual Compounding

$$i_{\text{eff}} = (1 + r/m)^m - 1$$

r = Nominal annual interest rate

n = Number of compounding periods

i_{eff} = Effective annual interest rate



The amount of interest earned if the bank is willing to compound more frequently than yearly is given above. Thus we could have computed your $\$6000$ account's value using:

NOMINAL

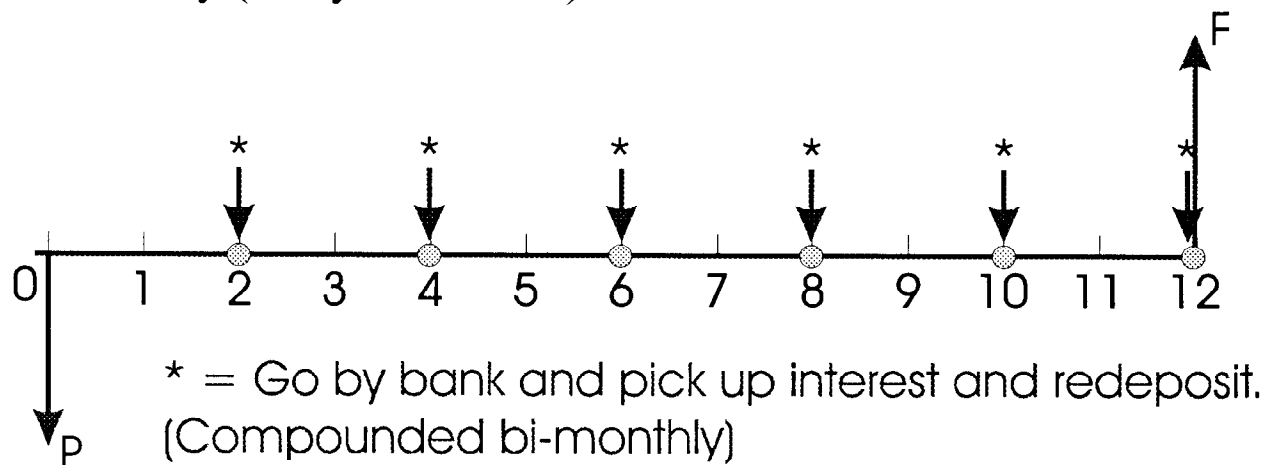
$$\begin{aligned}i_{\text{effective}} &= (1 + \text{rate}/m \text{ payment periods})^m - 1 \\ &= (1 + 0.06/4 \text{ payment periods})^4 - 1 \\ &= 0.06136\end{aligned}$$

Thus your account would be worth $\$6000(1 + i_{\text{effective}}) = \$6000(1.06136) = \$6368.18$, which is what we found before.

Also, at the end of 7 years your account would be worth

$$\begin{aligned}\$6000(1 + i_{\text{effective}})^7 &= \$6000(1 + 0.06136)^7 = \$6000(1.51719) = \\ &= \$9103.12\end{aligned}$$

Now what if you could find a bank which would compound your interest bi-monthly (every 2 months?)



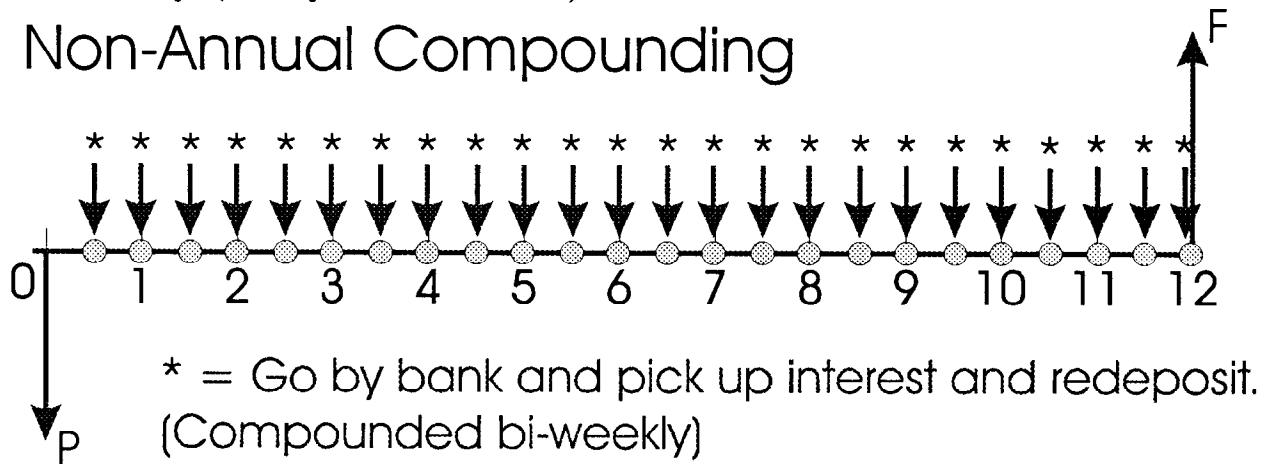
Then a \$6000 deposit invested at a nominal rate of 6% compounded bi-monthly would be worth (at the end of one year:

$$\begin{aligned}i_{\text{effective}} &= (1 + \text{rate}/m \text{ payment periods})^m - 1 \\ &= (1 + 0.06/6 \text{ payment periods})^6 - 1 \\ &= 0.06152\end{aligned}$$

$$F = \text{Future Value} = \$6000(1+0.06152) = \$6369.12$$

Well, what if I am willing to compound your nominal interest rate of 6% bi-weekly (every two weeks?)

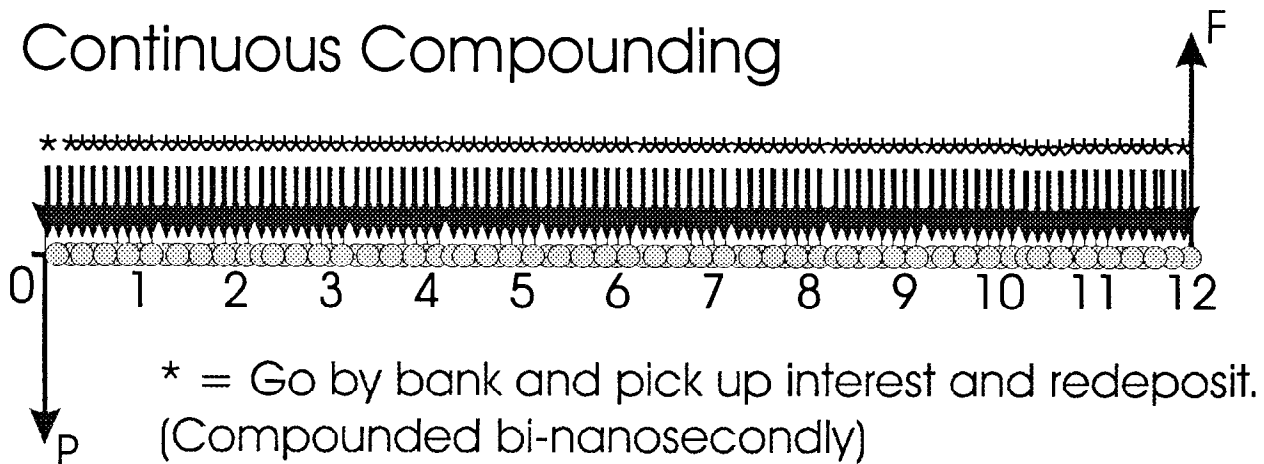
Non-Annual Compounding



$$F = \$6000(1+(0.06/26)^{26} - 1)) = \$6370.58$$

And finally, what if I am willing to compound your nominal interest rate CONTINUOUSLY???

Continuous Compounding



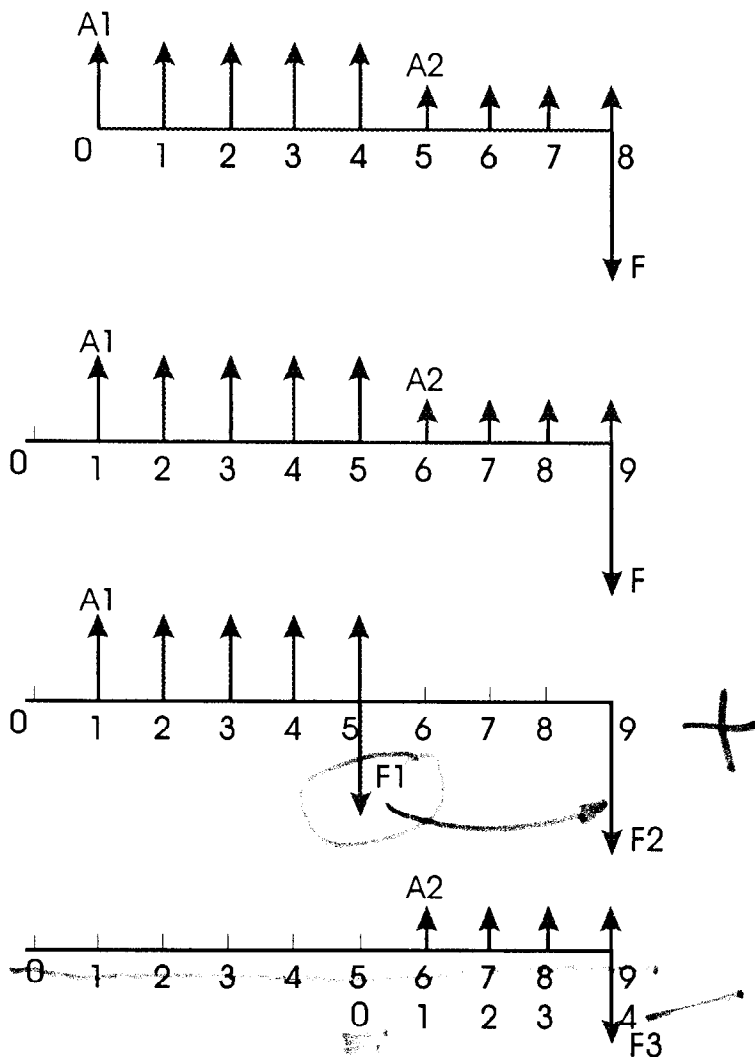
Then the effective interest rate is given by $i_{\text{effective}} = e^r - 1$

$$\text{For us, } i_{\text{effective}} = e^r - 1 = 2.718^{0.06} - 1 = 0.0618365$$

and your account value becomes $\$6000(1+0.0618365) = \6371.02

Mixed cash flows:

Question: If you invest \$600 today (!!!) and \$600 more at the end of each year for a total of 5 years, and then deposit \$200 at the end of the next 4 years, how much money will have accrued at the time of your last deposit at 6%?



$$F1 = A(F/A, 6\%, 5 \text{ yr}) = \$600(5.6371) = \$3382.26$$

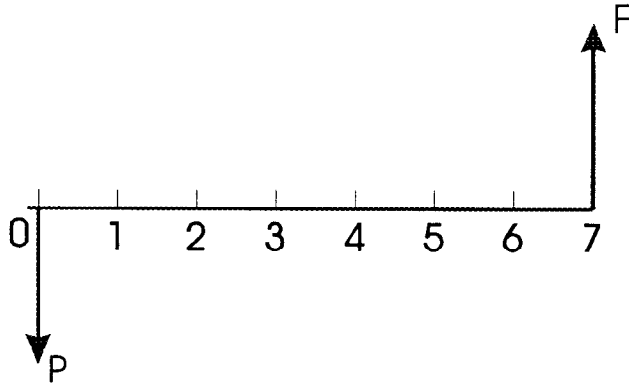
$$F2 = F1(F/P, 6\%, 4 \text{ yr}) = \$3382.26(1.2625) = \$4270.10$$

$$F3 = A(F/A, 6\%, 4 \text{ yr}) = \$200(4.3746) = \$874.92$$

Thus $F = F_2 + F_3 = \$5145.02$

Continuous Compounding:

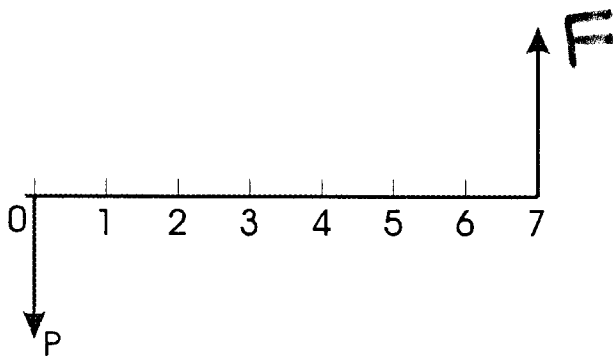
FIND: The Future value of an investment given the Present value for



Continuous Compounding:

How much would \$20,000 invested today be worth at a nominal annual interest rate of 6% compounded continuously for 7 years? Note that in these equations the interest rate is no longer labeled i - rather it is given the name r to show continuous compounding.

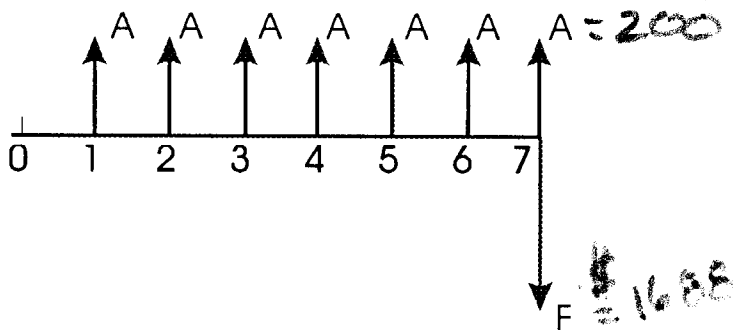
$$F = P(F/P, r\%, n) = \$20,000(e^{r \cdot n}) = \$20,000(e^{0.06 \cdot 7}) = \$30,439.23$$



FIND: The Present value needed to yield a Future value using Continuous Compounding:

How much would have to be invested today to yield \$30,439.23 under continuous compounding for 7 years if the nominal annual interest rate is 6%?

$$P = F(P/F, r\%, n \text{ years}) = \$30,439.23(e^{-rn}) = \$30,439.23(e^{-0.06*7}) = \$20,000$$



FIND: The future value of a series of 7 uniform deposits of \$200 each at the end of years 1 through 7 (standard form), if the bank pays 6% annual nominal interest rate compounded continuously.

$$F = A(F/A, r\%, n) = \$200(e^{rn} - 1)/(e^r - 1) = \$200(e^{0.06*7} - 1)/(e^{0.06} - 1) = \$200(8.440988) = \$1688.20$$

Book Value (All Methods of Depreciation)

Book value = Initial cost of the depreciable asset less the Sum of all depreciation taken up to the present time.

Straight Line Depreciation

$$D_j = (C - S_n) / n$$

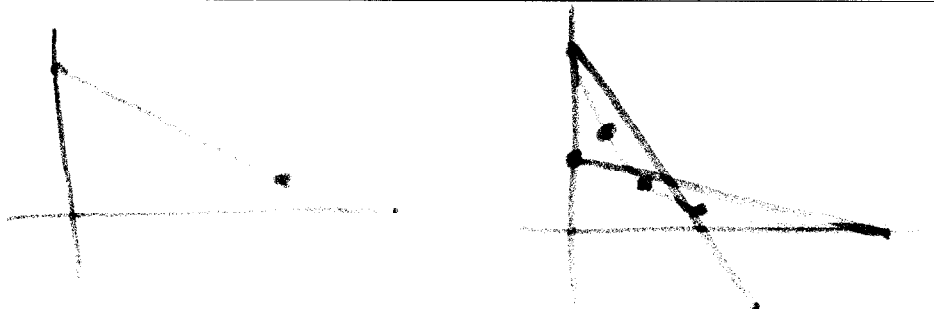
Where D_j = depreciation taken in year j , C = Initial cost of the depreciable asset used to make money, S_n = expected salvage value of the asset at the end of year n , and n = expected life of the asset (years.)

So, say you purchase a \$30,000 machine to make income with this year. You think the machine will be useful for 7 years before you will junk it. Also, at that time you expect to be able to sell it for \$2500.

$$\text{Then } D_1 = (\$30,000 - \$2,500) / 7 = \$3928.57$$

This is the amount that you can depreciate from your taxes at the end of each year, and also the amount of money you should take from your profits to buy a new \$30,000 machine when this one wears out. (Perhaps a little less because of the interest it makes.) Thus, the resulting BOOK VALUE at the end of each year will be:

End of Year	Depreciation	Book Value
0	0	\$ 30,000.00
1	\$ 3,928.57	\$ 26,071.43
2	\$ 3,928.57	\$ 22,142.86
3	\$ 3,928.57	\$ 18,214.29
4	\$ 3,928.57	\$ 14,285.72
5	\$ 3,928.57	\$ 10,357.15
6	\$ 3,928.57	\$ 6,428.58
7	\$ 3,928.57	\$ 2,500.01 (salvage)



Accelerated Cost Recovery System (ACRS)

$D_j = (\text{a factor from the table}) \text{ Original Total Cost of the asset}$

Unlike the straight line method, you do not need to estimate salvage. You get to depreciate the entire cost. The property is classed as being of a certain class life, either 3, 5, 7, 10, 15, and 20 years. The reference manual only gives tables for 3 through 10 years. The property class will have to be stated in the problem, since the guidelines are not listed in the manual.

Say you purchase a \$36,000 machine to make income with, and that the life has been classed by the government for income tax purposes as being in the 5 year life class. What is the depreciation you can take during each year, and what is the book value at the end of each year?

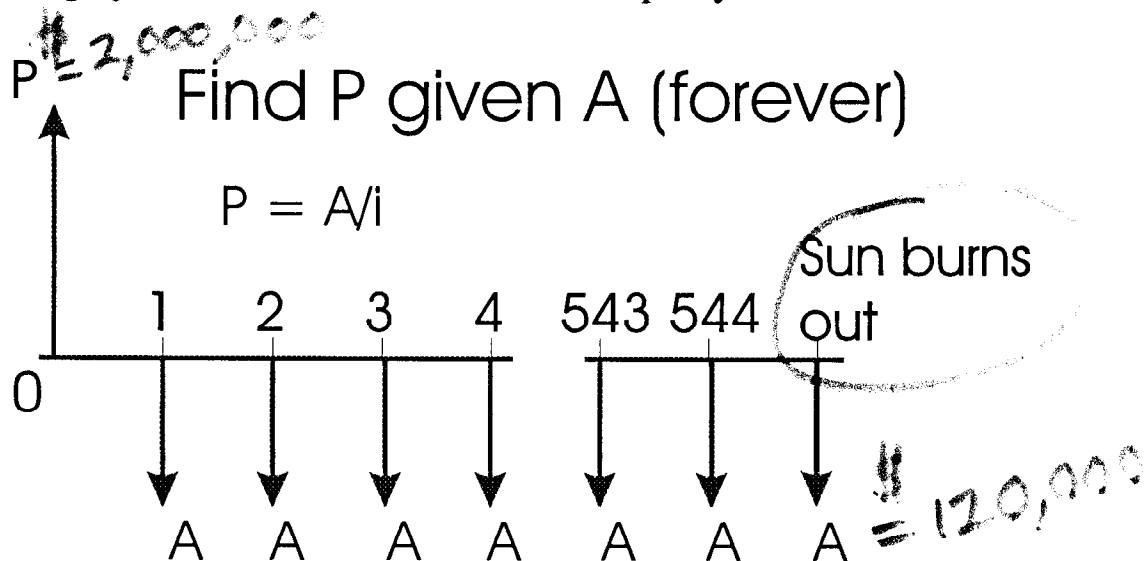
Year	Depreciation	Book Value
0		\$ 36,000
1	$\checkmark 0.20 * \$36,000 = \$ 7,200$	\$ 28,800
2	$\checkmark 0.32 * \$36,000 = \$ 11,250$	\$ 17,280
3	$\checkmark 0.192 * \$36,000 = \$ 6,912$	\$ 10,368
4	$\checkmark 0.115 * \$36,000 = \$ 4,140$	\$ 6,228
5	$\checkmark 0.115 * \$36,000 = \$ 4,140$	\$ 2,088
6	$\checkmark 0.058 * \$36,000 = \$ 2,088$	\$ 0
	Total Depreciation Above = \$ 36,000	

The method uses the \$200% declining balance method, permitting you to depreciate the asset much more quickly during its early life, but permitting you only half of the first year's depreciation during the first year that you purchased the asset, regardless of when during the year you purchased it. Thus if you purchased the asset in December, you still get

one-half year's depreciation the first year. Also, if you purchase it in February, you unfortunately only get 1/2 of a year's depreciation that year. Further, at some time later in the life of the asset, it would be to your advantage to switch from the 200% declining balance method to straight line method, which is permitted by the IRS, and the numbers listed in the ACRS table do this automatically.

Capitalized Costs

Project I) How much money should we ask the taxpayers to give us if we need to repair a dam for the next million years, and the yearly repair cost will be \$120,000 a year, and assuming that the money gleaned from the taxpayers will be invested at 6% per year?



So, $P = A / i = \$120,000/\text{year} / 0.06/\text{year} = \$2,000,000$

Why it works:

If you invest \$2,000,000 at 6% compounded yearly for one year, at the end of that year you will have your original \$2,000,000 plus $\$2,000,000 * 0.06 = \$120,000$. Thus you can take the \$120,000 interest you made out

of the bank to make the repairs at the end of the year, and still have \$2,000,000 remaining in the bank to earn the next year's needed \$120,000, forever and ever.

Project II) Now what if we need \$10,000 for repairs each month? At 6% nominal interest rate, compounded monthly, how much do we need to extract from the taxpayers?

$$P = A / i = \$10,000/\text{month} / (0.06/\text{year})(1 \text{ year}/12 \text{ months}) \\ = \$10,000/\text{month} / 0.005/\text{month} = \$2,000,000 \text{ Total}$$

Why it works: If you invest \$2,000,000 at a nominal interest rate of 6% per year and can find a bank kind enough to compound your interest monthly, then you are actually getting $0.06/12 = 0.005/\text{month}$, which is a lot better than 6% a year, and you will thus earn $0.005/\text{month} * \$2,000,000 = \del{\$100,000} 10,000/month to make your repairs.$

Note that the Project I bank was only willing to pay us 6% compounded yearly, so we had to wait until the end of the year to take out \$120,000, whereas the Project II bank was willing to compound monthly, permitting us to take out \$10,000 a month - a much better deal for us. As a matter of fact, the second bank is giving us an effective interest rate of:

$$i \text{ effective} = (1 + r/m)^m - 1 = (1 + 0.06/12)^{12} - 1 \\ = 0.0616778 = 6.16778\% \text{ per year}$$

Well heck. If the second bank is a better deal, let's take Project I to him instead. How much will we have to put in his bank to take out \$120,000 at the end of each year, forever?

$$P = \$120,000/\text{year} / 0.0616778 \text{ effective}/\text{year} = \$1,945,594.70$$

which saves us $\$2,000,000 - \$1,945,594.70 = \$54,405.31$ on our Project I bond issue.

But wait! Bank 3 says they will give us 6% nominal interest rate compounded weekly:

$i \text{ effective} = (1 + 0.06/52)^{52} - 1 = 6.17998 \%$, such that

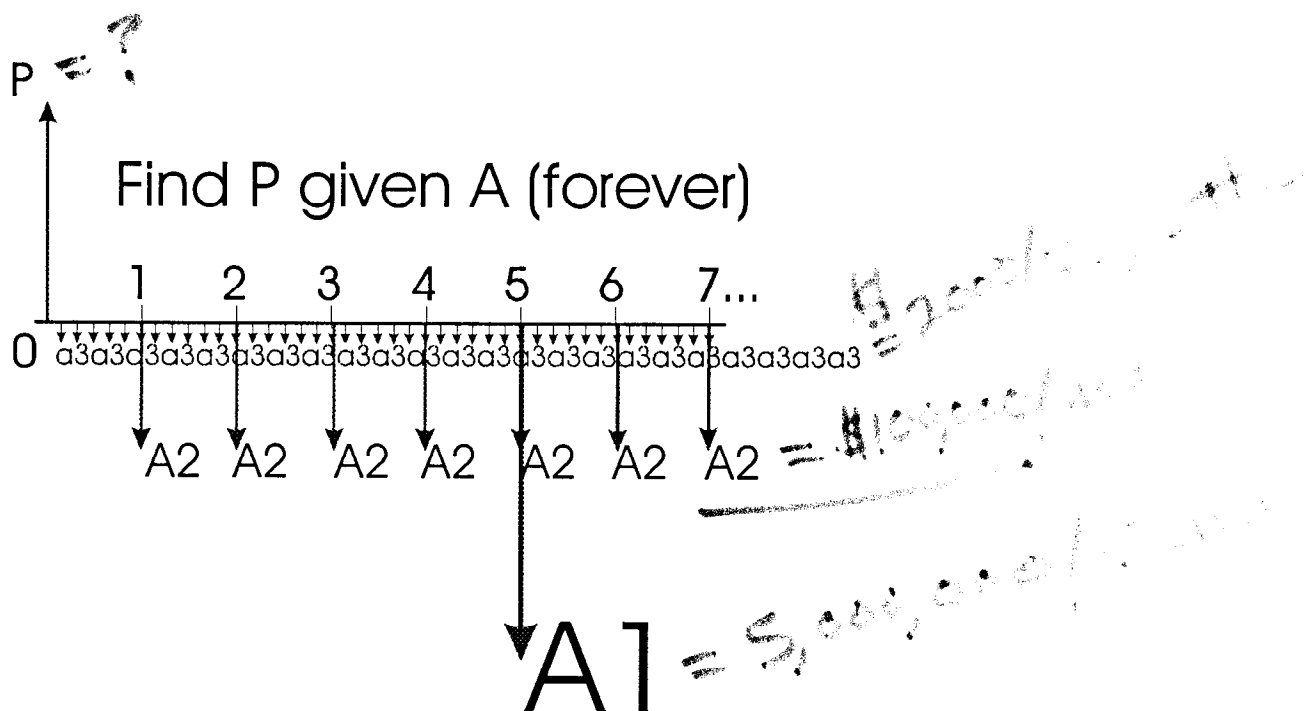
$$P = \$120,000/\text{year} / 0.0617998 = \$1,941,753.20$$

And if we can find a bank which pays 6% nominal interest rate compounded continuously:

$i \text{ effective} = e^r - 1 = e^{0.06} - 1 = 0.0618365$, such that

$$P = \$120,000/\text{year} / 0.0618365 = \$1,940,000$$

Project III) A&M has been told that they may win their bid to house the Bush Library, if they can guarantee that they can fund the upkeep and repairs in perpetuity (i.e. forever.) The library is expected to cost \$2000 every two months for light bulbs, plus \$100,000 every 12 months for minor repairs, plus an additional \$5,000,000 every 5 years for a complete refurbishing of the facility. How much must be deposited today to guarantee the funding of these expenses? Assume a nominal interest rate of 6%, compounded monthly.



Now the easiest way to work problems like this is to treat each annuity value as a future value needed somewhere down the road and change it into an equivalent annuity based on the compounding period of your loan, or rate. For example, how much money would you have to invest each month (use the compounding period) to have \$100,000 at the end of 12 months? Easily found by taking the future value needed (\$100,000) and transforming it into an equivalent monthly annuity:

$$i \text{ per payment period} = 0.06/12 = 0.005\%$$

$$\begin{aligned} A2 &= F2(A/F, i \text{ per payment period}, n) \\ &= \$100,000 \text{ (Use A/F column, 0.005 table, 12 payments)} \\ &= \$100,000 * 0.0811 = \$ 8110 \end{aligned}$$

This says that if you will invest \$8110 dollars each month as an annuity (you must follow the rules - first deposit goes in at the end of the first month,) for a total of 12 months, you will have \$100,000 at the end of the 12 month period. You can then withdraw this to make your necessary

repairs, and then continue depositing \$8110 each month forever, and you will always be able to withdraw \$100,000 at the end of every year.

Now let's change the a3 into a monthly (the bank's compounding period) annuity. Setting the value needed (\$2000 every two months) to the Future value needed, and realizing that we are paid 0.005% per period, and that we have two periods to get the money:

$$\begin{aligned} A3 &= F3(A/F, i \text{ per payment period}, n) \\ &= \$2,000 \text{ (Use A/F column, 0.005 table, 2 payments)} \\ &= \$2,000 * 0.4988 = \$997.60 \end{aligned}$$

This says that if you will invest \$997.60 dollars each month as an annuity (you must follow the rules - first deposit goes in at the end of the first month,) for a total of 2 months, you will have \$2,000 at the end of the 2 month period. You can then withdraw this to make your necessary repairs, and then continue depositing \$997.60 each month forever, and you will always be able to withdraw \$2,000 at the end of every 2 month period.

Now let's change the A1 into a monthly (the bank's compounding period) annuity. Setting the value needed (\$5,000,000 every 60 months) to the Future value needed, and realizing that we are paid 0.005% per period, and that we have 60 periods to get the money:

$$\begin{aligned} A1 &= F3(A/F, i \text{ per payment period}, n) \\ &= \$5,000,000 \text{ (Use A/F column, 0.005 table, 60 payments)} \\ &= \$5,000,000 * 0.0143 = \$71,500 \end{aligned}$$

This says that if you will invest \$71,500 dollars each month as an annuity (you must follow the rules - first deposit goes in at the end of the first

month,) for a total of 60 months, you will have \$5,000,000 at the end of the 60 month period. You can then withdraw this to make your necessary repairs, and then continue depositing \$71,500 each month forever, and you will always be able to withdraw \$5,000,000 at the end of every 60 month period.

Thus, in total, you have been asked to scrape up a total of \$80,587.60 each and every month forever, to fund the library. If instead I wish to make a single deposit today, I can find this by:

$$P = A / i = 80,587.60/\text{month} / 0.005/\text{month} = \$ 16,117,520$$

Bond Value

Bond value equals the present worth of the payments the purchaser or holder of the bond receives during the life of the bond. Bonds are sold for some number of years, n , during which the seller normally pays some monthly or yearly amount to the purchaser. Then, at the end of n years, or months, the seller repays the original bond cost back to the buyer. The Bond Yield equals the computed interest rate of the bond value when compared with the bond cost.

The owner of a bond is normally paid back for his purchase in two ways. First they are paid a series of periodic payments monthly, or quarterly, or semi-annually, or annually, until the bond is retired. These payments are usually r (the bond rate = nominal interest rate) times the cost of the bond. Thus if you buy a 6% bond for \$1000 and are paid quarterly, you would receive $\$1000 * (0.06/4) = \$15/\text{quarter}$ every quarter until the bond became due. When the bond is retired you would then be paid back the cost of the bond. Thus the Bond Value of the bond (to you) will be the Present value of the monthly (or quarterly, or yearly) annuity payments, plus the Present

value of the final repayment F , both based on the desired rate of interest you require.

For example, determine how much you would be willing to pay (or the "Bond Value") of a 10 year bond sold for \$1000 and paying a nominal interest rate of 6%. Assume that the bond pays semi-annually. Also assume that you will not buy the bond as an investment unless it actually yields 8%.

The bond's periodic payments will be $\$1000 (0.06 / 2) = \30 every 6 months. Thus the value of this stream of income to you, since you want 8% interest is:

$$P = A (P/A, i \text{ desired}, n \text{ payments})$$
$$= \$30 \text{ (Use P/A column, } \cancel{3\%} \text{ } 4\% \text{ per payment period, 20 payments)}$$
$$= \$30 (13.5903) = \$ 407.71$$

Now the final repayment of the value of the bond will be \$1000, and the present value of this, since you want 8% on your money, will be:

$$P = F (P/F, i \text{ desired}, n \text{ payments})$$
$$= \$ 1000 \text{ (Use P/F column, } \cancel{3\%} \text{ } 4\% \text{ per payment period, 20 payments)}$$
$$= \$1000 (0.4564) = \$456.40$$

Thus to actually return 8% interest you would have to pay no more than $\$407.71 + \$ 456.40 = \$ 864.11$ for the bond, which would be its face value to you. Note that the seller of the bond might suggest that the face value of his bond should be based on his 6% nominal interest rate. In that case his computations for what he would call the face value of the bond would substitute 3% per payment period for the 4% values you used above.

The Bond Yield equals the computed interest rate of the bond value when compared with the bond cost. Thus if you purchased the bond discussed above for \$864.11, the Bond Yield would indeed be 8%. If, however, someone wanted to cash in that bond bad enough, and asked you if you would buy it for \$656 cash on the barrel head, you would probably say, well, yes, I guess I might, possibly. Then to calculate the actual yield of your new bond, you have:

$$P = \text{Value of bond returned in 20 periods (P/F, } i \text{ yield, 20) + } \\ \text{(Periodic payments (A)(P/A, } i \text{ yield, 20))}$$
 where the Value of the bond to be returned = \$1000, and
 the Periodic payments were \$30 each payment period.

Thus
$$P = \overset{F}{\$1000}(\text{P/F, } i \text{ yield, 20) + } \overset{A}{\$30}(\text{P/A, } i \text{ yield, 20)}$$

Now in the real world this might be rather messy, and you would have to resort to a trial and error method to solve for i yield. However, on the FE exam it will be a snap, since they will give you 5 possible answers, each of which you quickly plug in and see if that's the answer. For example, say they give you the 5 choices a) ~~4%~~, b) ~~6%~~, c) ~~8%~~, d) 12%, e) 16% nominal rate per year.

Well, 4% and 6% and 8% sure aren't correct, since you know 8% was the yield if you paid the full \$864.11. So, let's try 12%:

$$P = \overset{F}{\$1000}(\text{P/F, } i \text{ yield, 20) + } \overset{A}{\$30}(\text{P/A, } i \text{ yield, 20)}$$

\$ 656 $\stackrel{F}{=}?$ $\overset{A}{\$1000}(\text{P/F, } 0.06, 20) + \overset{A}{\$30}(\text{P/A, } 0.06, 20)$

\$ 656 $\stackrel{F}{=}?$ $\overset{A}{\$1000}(0.3118) + \overset{A}{\$30}(11.4699)$

\$ 656 $\stackrel{F}{=}?$ $\overset{A}{\$311.80} + \overset{A}{\$344.10} = \overset{F}{\$ 655.90}$ CLOSE ENOUGH! ✓

Rate of Return

The Minimum Attractive Rate of Return is the highest rate of return that you can obtain in alternative investments, and hence the minimum rate that you would accept. The actual rate of return you receive on an investment would then hopefully be greater than this MARR. The actual rate of return could be computed by setting the Present Worth Benefits = Present Worth Cost, or the Uniform Annual Benefits = Uniform Annual Costs, or the Future Worth Benefits = Future Worth Costs. All would be equivalent. By setting the benefits = costs, you can compute an interest rate which would actually cause them to be equal, thus telling you what interest rate you will be receiving, and whether that rate exceeds your MARR or not. This rate is called the internal rate of return.

Example: Find the internal rate of return for the following cash flow:

End of Year	Receipt/Disbursement
0	-\$1000
1	\$250.50
2	\$250.50
3	\$250.50
4	\$250.50
5	\$250.50

Answers: (a) 2% (b) 4% (c) 6% (d) 8% (e) 10%

Set PWB = PWC

$$\$250.50 (P/A, \text{IRR}\%, 5) = \$1000$$

$$(P/A, \text{IRR}, 5) = \$1000/\$250.50 = 3.992$$

(Look in P/A column, look in 6% table, look on 5 year row) = 3.992?

Nope, too big = 4.2124. Try the 8% table:

(Look in P/A column, look in 8% table, look on 5 year row) = 3.992?
Yep, that's it, or close enough. Answer is d.

Note I started in the middle of the possible answers, since that way I won't have to try anymore than two others if (c) is not correct.

Example: Determine the internal rate of return on a new piece of equipment that will cost \$25,000, and will increase productivity by \$5050 each year for 5 years. Further, the equipment will have a resale salvage value of \$5000 at the end of the 5 years. Answers: (a) 2% (b) 4% (c) 6% (d) 8% (e) 10%

In this case there are two terms which contain the IRR%, such that the IRR cannot be directly found. However, since they give you 5 choices, just try them on until you get one that fits. Start with the middle one:

Set Present Benefit = Present Cost

$$A (P/A, IRR\%, n) + F (P/F, IRR\%, n) = \$25,000$$

$$\$5050 (\text{Look in P/A, IRR}\%, 5) + \$5000 (\text{Use P/F, IRR}\%, 5) = \$25,000$$

Try 6%:

$$\$5050 (\text{Look in P/A, 6}\%, 5 \text{ yr}) + \$5000 (\text{Use P/F, 6}\%, 5 \text{ yr}) = \$25,000?$$

$$\$5050 * 4.2124 + \$5000 * 0.7473 = \underline{\$25,009} \text{ Close enough.}$$

Answer is (c)

Break-Even Analysis

The break-even point is found by varying one of the decision variables in a study, until it causes two alternative investments to become equal. For example, you might have two pumps available for purchase, each costing different amounts for initial cost and yearly costs, wherein one is cheaper if you only need to pump small amounts, and the other is cheaper if you

have to pump large quantities. Then you could find a break-even point by varying the amount to be pumped until the costs of the two pumps were equal.

Example: Assume that a pump is needed to remove water from excavations. Further, assume that Pump A initially costs \$1,800, costs \$1.10 per hour to operate, requires \$360/year in maintenance and will have a salvage value of \$600 at the end of its 4 year life. Pump B has an initial cost of \$550, costs \$2.35 per hour to operate, requires no maintenance, and has no salvage value at the end of 4 years. Use an interest rate of 6%.

Then, the annual equivalent fixed costs will be:

Initial Cost:

Pump A: $A = \$1,800 (A/P, 6\%, 4 \text{ years}) = \$1,800 * 0.2886 = \$519.48$

Pump B: $A = \$550 (A/P, 6\%, 4 \text{ years}) = \$550 * 0.2886 = \$158.73/\text{year}$

Salvage Value:

Pump A: $A = \$600 (A/F, 6\%, 4 \text{ years}) = -\$600 * 0.2286 = -\$137.16/\text{year}$

Pump B: \$0/year

Maintenance Cost:

Pump A: \$360/year

Pump B: \$0

Variable Yearly Costs:

Pump A: \$1.10/hour * X hours/year

Pump B: \$2.35/hour * X hours/year

Then setting the costs of the two alternatives equal to each other, we can solve for the number of hours of pumping which divides when each pump should be selected:

Pump A annual equivalent cost = Pump B annual equivalent cost

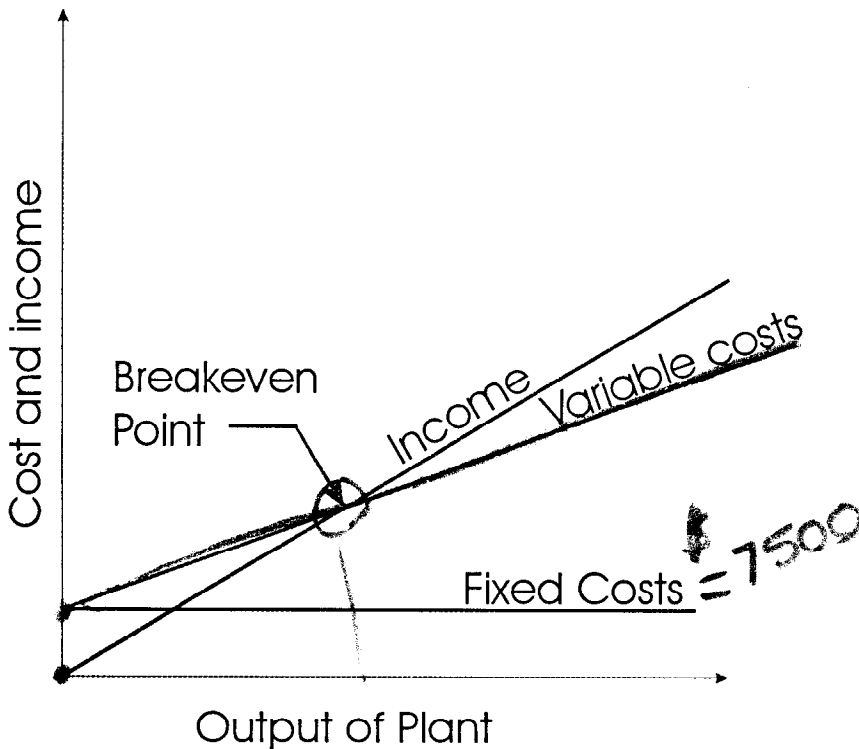
$\$519.48 - \$137.16 + \$360 + \$1.10X = \$158.73 - \$0 + \$0 + \$2.35X$

$X = 486.3$ hours per year.

Thus if the pumping requirements of the next job are less than 486.3 hours, Pump B would be the better selection. However, if the pump will operate more than 486.3 hours per year, Pump A would be better.

A second use of break-even analysis is to determine at what level a manufacturing plant must operate in order to break even, considering its fixed and variable costs, and profit.

Example: A company is trying to determine at what level they must operate their plant just to break even. Their fixed costs are \$7500/month and their variable costs are \$3.00 per unit. Revenue is expected to be \$10/unit.



As shown above, the Fixed costs are plotted as \$7500/month, and on top of this the Variable costs of \$ 3.00/unit are added. Then the income is plotted, started from zero since if there is no production the only cost will be the fixed monthly cost.

Solving for X, the number of units required to break even:

Costs = Benefits

$$\$7500 + \$3.00X = \$ 10.00X$$

$$X = 1071 \text{ units}$$

Thus we must make at least 1071 units, just to break even.

Payback Period

The payback period is commonly defined as the time required to recover an initial investment from the net cash flow from that investment. It can include the effect of interest, or often simply ignores the interest. Thus, you might say that an investment of \$10,000, which pays back \$5000/year at the end of the next two years has a payback period of 2 years.

Example: Find the payback period for the following investment, assuming an interest rate of 6%:

End of Year	Investment A
0	- \$10,000
1	\$ 0
2	\$ 2,000
3	\$ 4,000
4	\$ 4,000
5	\$ 2,270
6	\$ 4,000

Thus at the end of the 0th year, we are \$ 10,000 in the hole.

For year 1, we add an additional present value of $\$0 * (P/F, 6\%, 1) = \0

For the end of year 2, we would add $\$2000 * (P/F, 6\%, 2) = \1780

F

1790

At the end of year 3, we would add $\$4000 * (P/F, 6\%,3) = \3358 -

At the end of year 4, we add $\$4000 * (P/F, 6\%,4) = \3168 -

At the end of year 5, add $\$2270 * (P/F, 6\%,5) = \1696 -

At the end of year 6, add $\$4000 * (P/F, 6\%,6) = \2820 -

Thus, adding the additional present values year by year, we see that after the 5th year, we would have our money back, including the effect of interest and the fact that someone else was enjoying our money instead of us.

If the quiz problem does not give an interest rate, then they are using the more common definition of payback period. For the problem above, excluding interest, the payback period would be the end of the 4th year.

Inflation

Handwritten note: $d = i + f + (i * f)$

The reference manual uses the equation $d = i + f + (i * f)$ to handle inflation, where

d = the stated interest rate paid by the bank, or paid on the bond, or stated by the lender. This is the "combined" interest rate - i.e. for the lender it combines the real rate of return he desires, with the inflation rate. It is also known as the "market" rate, since this is usually what the market charges for money.

f = the inflation rate.

i = the "real" interest rate - i.e. the "real" increase in purchasing value the lender is getting by loaning his money to the borrower.

Example: What is the actual present value (to you) of a bond which pays \$6000/year at the end of each year for 10 years, if the purchase price of \$20,000 would be paid back at the end of the 10th year. Assume that you

could get 6% interest on other investments, and that this is your MARR (minimum acceptable rate of return.)

$$\begin{aligned} P &= A(P/A, IRR\%, n) + F(P/F, IRR\%, n) \\ &= \$6000(\text{Use P/A column, 6\%, 10 years}) + \$20,000(P/F, 6\%, 10 \text{ yr}) \\ &= \$6000(7.3601) + \$20,000(0.5584) \\ &= \$ 55,328.60 \end{aligned}$$

Thus you should not be willing to pay more than this for the bond. This is the "value" of the bond (to you.)

Now what if you think that the inflation rate for the next 10 years will be 4%? How will this influence the value of the bond? Well, first you will realize that the dollars with which you will be repaid will not have the same purchasing power, or value. Since you will still want a "true" rate of return of 6%, you will not be willing to pay as much for the bond. Its value will decrease. What is its value (to you?)

(a) \$44,030 (b) \$44,580 (c) \$55,330 (d) \$58,620 ~~(e)~~ ^{e.} \$61,424

Using d combined = $i + f + (i*f) = 0.06 + 0.04 + (0.06*0.04) = 0.1024$

Now to get a quick and dirty answer, let's assume this came out $d = 10\%$. This would allow us to use the tables (there aren't any 10.24% tables.)

$$\begin{aligned} P &= A(P/A, i, n) + F(P/F, i, n) \\ &= \$6000(\text{Use P/A column, 10\%, 10}) + \$20,000(P/F, 10\%, 10) \\ &= \$6000(6.1446) + \$20,000(0.3855) \\ &= \$44,578 \end{aligned}$$

Now this looks like answer (b), but that must be a trick, since we aren't really using the correct interest rate. So my guess would be that answer (a) is the correct answer. But, let's check it.

$$\begin{aligned}
 P &= A(P/A, i, n) + F(P/F, i, n) \\
 &= A((1+i)^n - 1) / (i(1+i)^n) + F(1+i)^{-n} \\
 &= \$6000((1+0.1024)^{10} - 1) / (0.1024(1+0.1024)^{10}) + \$20,000(1+0.1024)^{-10} \\
 &= \$6000(6.0817) + \$20,000(0.37723) \\
 &= \$44,034.83 \text{ Yep. Answer is (a).}
 \end{aligned}$$

Example: A bank lends you \$10,000 today, to be repaid in a lump sum at the end of 10 years at a combined, market, stated interest rate of 10% compounded annually. Make sure you understand that this is the combination of what the bank hopes to gain, plus the influence of inflation. If the rate of inflation is 8%, what is the bank's true rate of return, and how much money will they really make in 10 years?

Remember that the reference manual calls "d" the "combined" rate of return, which means the interest rate found by adding both the "really made me some money = i" interest rate to the inflation rate = f. The "i" in the equation is the "real" interest rate. For example, if you made me a loan at d = 6% interest, and the inflation rate was f = 6%, you would actually not make any extra spendable money over the life of the loan. So what you must do as a lender is ask for a higher rate on the loan to give yourself some "real" rate of return "i." Be very sure to understand the terminology used here. It is very likely different with what you are accustomed.

Now let's see what the bank's "real" rate of return will be:

$i \cdot f$

Since $d = i + f + if$, we can solve for i :

$$i = (d-f)/(1+f) = (0.10 - 0.08) / (1 + 0.08) = 0.0185 = 1.85\%$$

So the bank's real rate of return will be 1.85%. How much money they actually made can be found by:

$$\begin{aligned} F &= P(F/P, i_{\text{real}}, n) = \$10,000(\text{Use } F/P \text{ equation, } 1.85\%, 10 \text{ years}) \\ &= \$10,000(1+i_{\text{real}})^n = \$12,011.86 \end{aligned}$$

So the bank will really make an additional \$2,011.86

Benefit-Cost Analysis

A benefit-cost analysis attempts to determine if the Benefits > Costs, or if Benefits/Costs > 1.0. This is normally done by computing either the Present Worth of the Benefits and comparing them with the Present Worth of the Costs, or by comparing their Annual Worths.

Example: A project is being considered wherein a small airfield will be built. Land for the project will cost \$350,000, construction costs will be \$800,000, and annual maintenance will be \$25,000/year. Annual benefits expected from this project = \$250,000 per year. Determine the Benefit-Cost ratio assuming a rate of 6% and a life of 20 years.

$$\begin{aligned} \text{Present Worth of Benefits} &= A(P/A, i, n) = \$250,000(P/A, 6\%, 20) \\ &= \$250,000(11.4699) = \$2,867,475 \end{aligned}$$

$$\begin{aligned} \text{Present Worth of Costs} &= \$350,000 + \$800,000 + A(P/A, i, n) \\ &= \$350,000 + \$800,000 + \$25,000(P/A, 6\%, 20) \\ &= \$1,436,475 \end{aligned}$$

Then $B/C = \$2,867,475/\$1,436,475 > 1.0$ and project should be considered.