FE Exam – Dynamics

Review

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Resource:

EIT Review Manual

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(Topic V: Dynamics

Resource:

"Fundamentals of Engineering
(FE)
Discipline Specific
Reference Handbook"

NCEES

pp 21–28 (Dynamics)

Engineering Mechanics:

"the study of forces and motions"

Statics (acceleration is zero)

Dynamics

Mech. of Materials (deformable bodies)

Dynamics

Kinematics (the geometry of motion)

Kinetics (dynamics)

Methods (tools)

$\sum \vec{F} = m\vec{a}$

Work/Energy methods

Impulse/Momentum methods

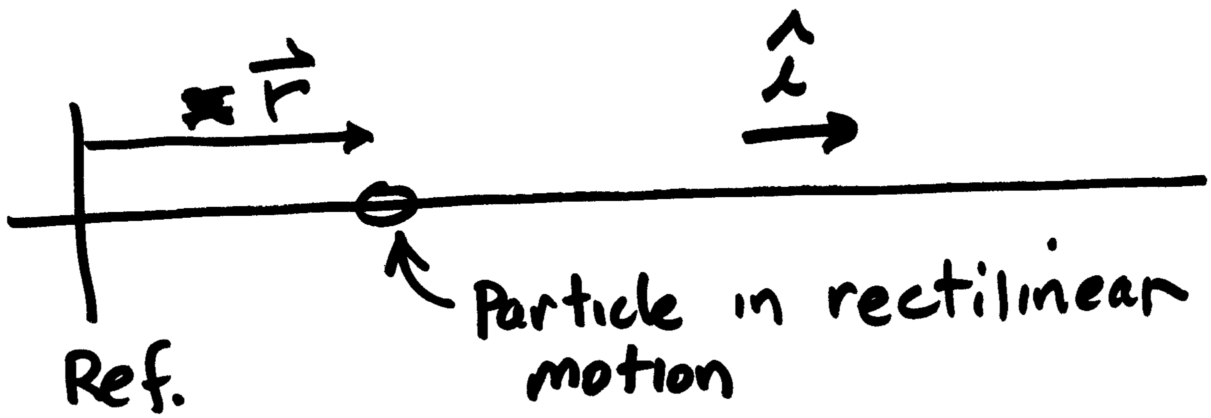
Particles & Systems of Particles

Rigid Bodies

Kinematics of Particles

Rectilinear Motion
Curvilinear Motion

Rectilinear Motion – along a straight line



let the position vector $\vec{r}$ be defined as above

Then

$$\vec{v} = \frac{d}{dt}\vec{r}$$

$$\vec{a} = \frac{d}{dt}\vec{v} = \frac{d^2}{dt^2}\vec{r}$$

since $\vec{r} = r\hat{i}$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{i} + r\cancelto{0}{\frac{d\hat{i}}{dt}} = \dot{r}\hat{i}$$

$$\left(\dot{r} \equiv \frac{d}{dt}r\right)$$

and

$$\frac{d\vec{v}}{dt} = \frac{d\dot{r}}{dt}\hat{i} + \dot{r}\cancelto{0}{\frac{d\hat{i}}{dt}}$$

(why is $\frac{d\hat{i}}{dt} = 0$?)

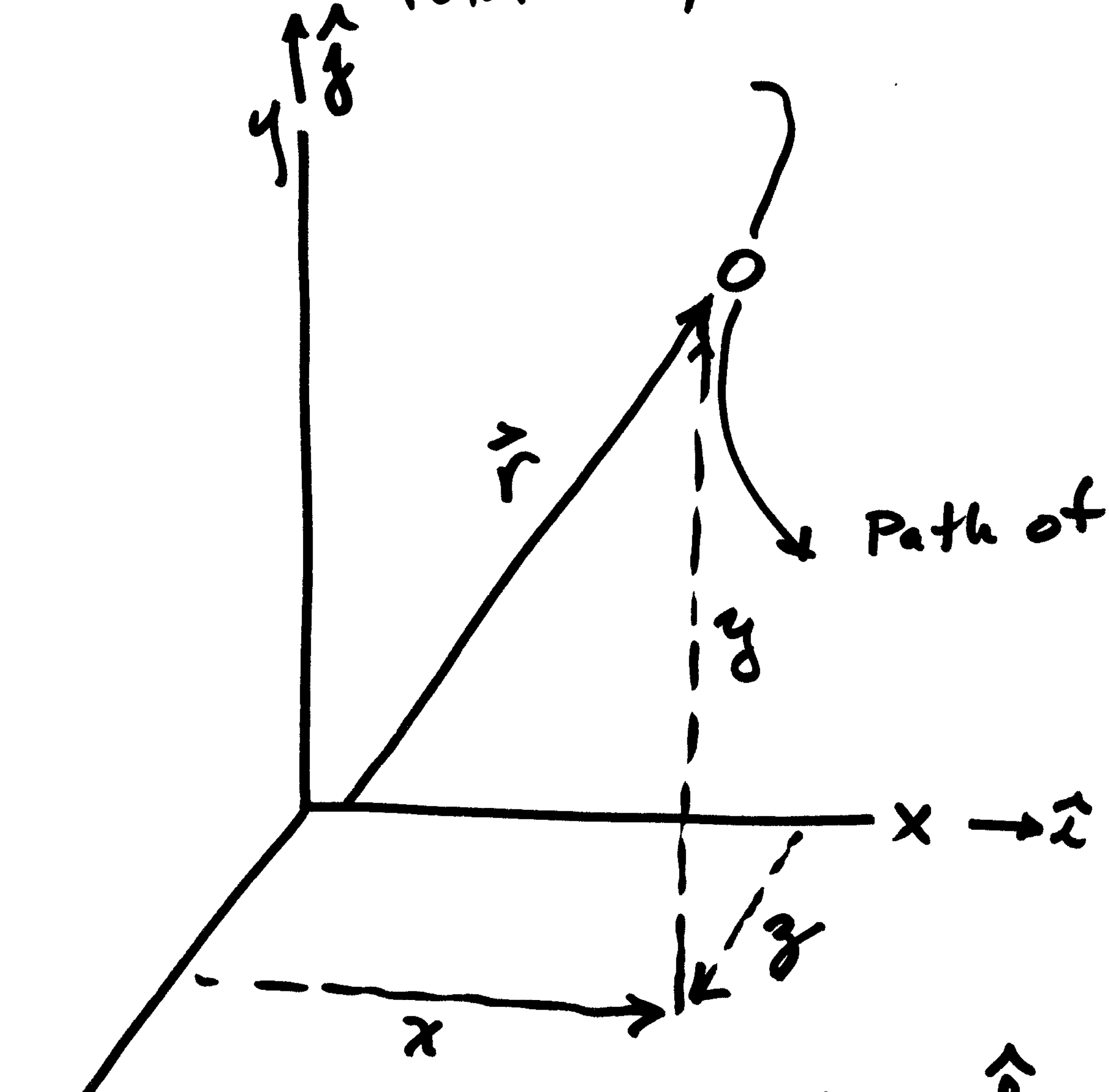
$$\vec{r} = r(t)\hat{i}$$
$$\vec{v} = \dot{r}(t)\hat{i}$$
$$\vec{a} = \ddot{r}(t)\hat{i}$$

obviously inverse (integral) relationships exist, also

$$v(t) = \int a(t)\,dt$$
$$r(t) = \int v(t)\,dt = \int\left[\int a(t)\,dt\right]dt$$

Curvilinear (General) Motion:

Rectangular (Cartesian) Coord's.
"Path" Coordinates (normal & tangential)
Polar or Cylindrical Coord's (radial & transverse)

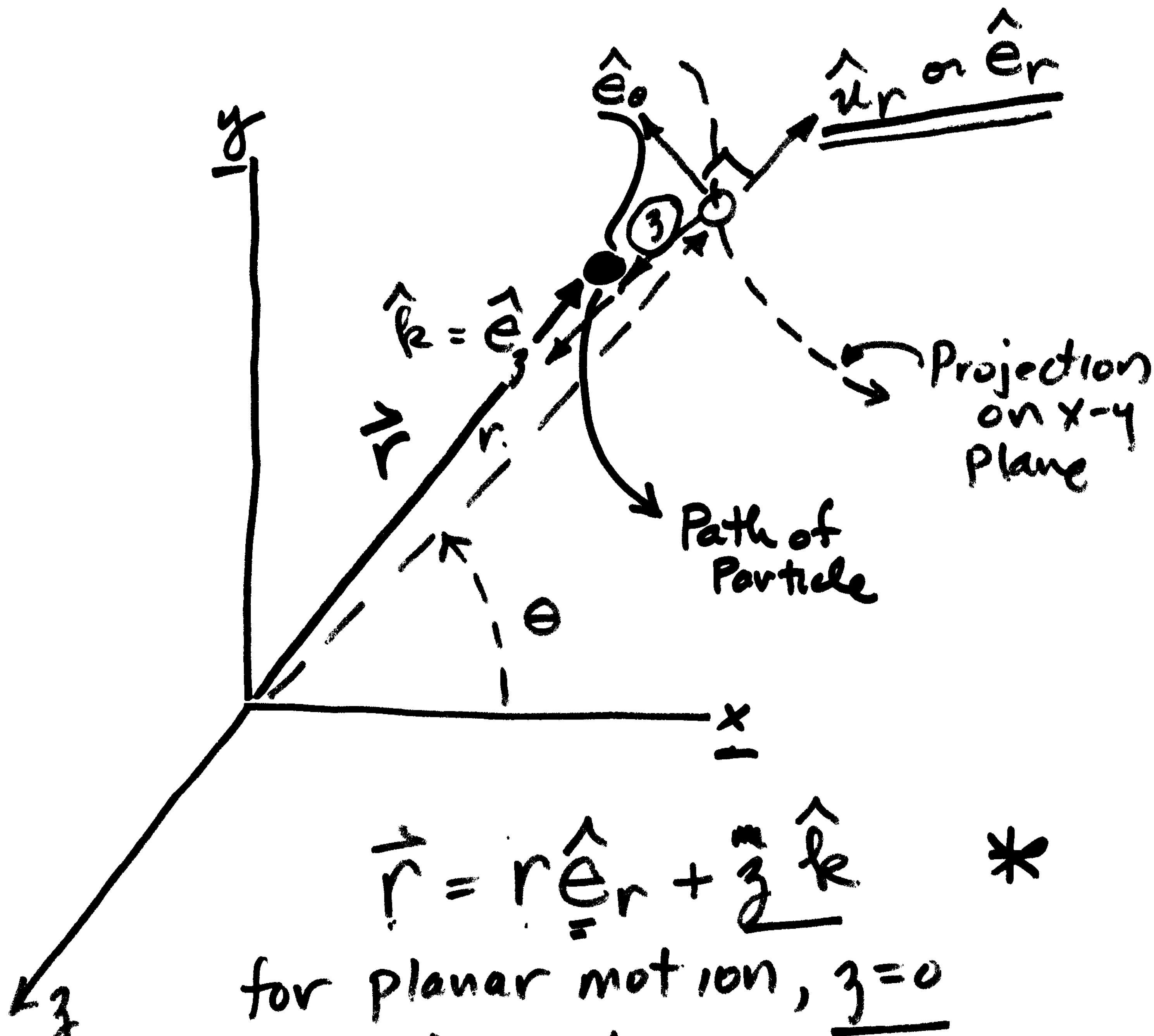


$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} + x\cancelto{0}{\frac{d\hat{i}}{dt}} + \dots$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

Cylindrical (3-D) or Polar (2-D) Coordinates



$$\vec{r} = r\hat{e}_r + z\hat{k} \quad *$$

for planar motion, $z=0$

$$\vec{r} = r\hat{e}_r$$

$$\vec{r} = r\hat{e}_r$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \underline{\dot{r}\hat{e}_r} + r\frac{d}{dt}\underline{\hat{e}_r}$$

...but $\frac{d}{dt}\hat{e}_r = \underline{\dot{\theta}\hat{e}_\theta}$

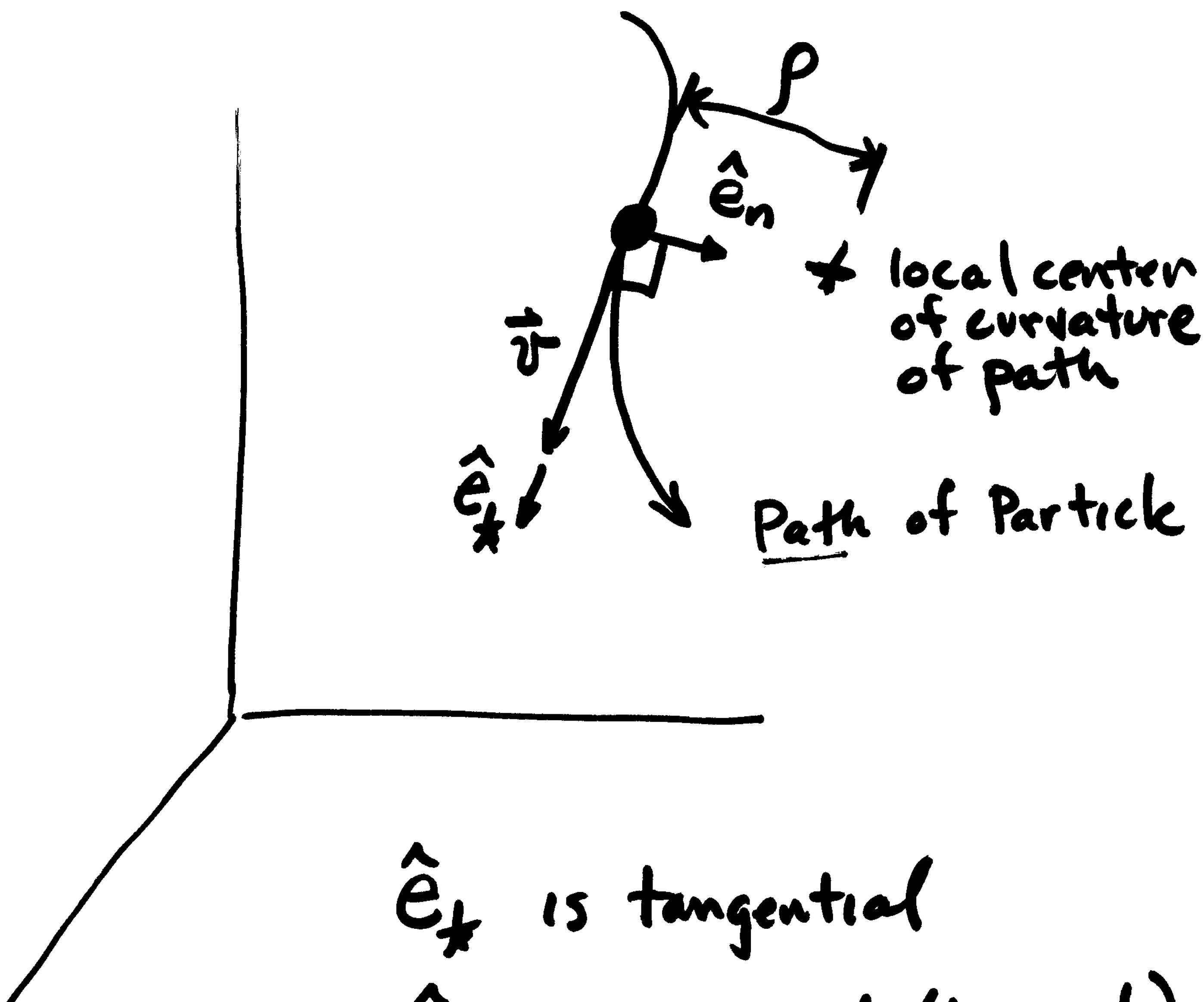
so $\vec{v} = \underline{\dot{r}\hat{e}_r} + \underline{r}\,\underline{\dot{\theta}\hat{e}_\theta}$

$$\vec{a} = \frac{d}{dt}\left(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right)$$

$$= \Big(\overbrace{\ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta} + \overbrace{\dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d}{dt}\hat{e}_\theta}\Big)$$

but $\frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\hat{e}_r$

$$\vec{a} = \underline{(\ddot{r} - r\dot{\theta}^2)}\hat{e}_r + \underline{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}\hat{e}_\theta$$



$\hat{e}_t$ is tangential

$\hat{e}_n$ is normal (inward)

$$\vec{v} = v\,\hat{e}_t$$

$$\vec{v} = v\,\hat{e}_t$$

$$\vec{a} = \frac{dv}{dt}\hat{e}_t + v\frac{d}{dt}\hat{e}_t$$

but $\frac{d}{dt}\hat{e}_t = \frac{v}{\rho}\hat{e}_n$ $\quad\left[\frac{1}{T}\right]$

$$\vec{a} = \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

ρ = instan. radius of curvatur[e]

Kinematics of Particles-
Sample Problems

1. (pp 14-4) (rectilinear motion)

Given $s = 20t + 4t^2 - 3t^3$ (m); t in sec

Find: initial velocity ($v(0)$)

$$v(t) = \frac{ds}{dt} = \underline{20 + 8t - 9t^2}$$

$$v(0) = \underline{20 \text{ m/s}} \quad \checkmark$$

2. Find $a(0)$

$$\underline{a(t) = \frac{dv(t)}{dt}} = 8 - 18t$$

$$a(\underline{0}) = 8 \text{ m/s}^2 \quad \checkmark$$

common error

Note: do not try to compute

$\times$

$$\frac{d}{dt}(\underline{\underline{v(0)}}) = \frac{d}{dt}(20 \text{ m/s}) = 0$$

$$\Rightarrow \underline{a(0) = 0} \quad \underline{\underline{\text{incorrect}}}$$

3. Find v_{max}

$$\underline{v(t) = 20 + 8t - 9t^2}$$

$$\underline{a(t) = 8 - 18t}$$

v_{max} occurs when $\underline{\frac{dv}{dt} = a = 0}$

$$a(t) = \underline{0 = 8 - 18t'}$$

$$t' = \frac{8}{18} \text{ sec}$$

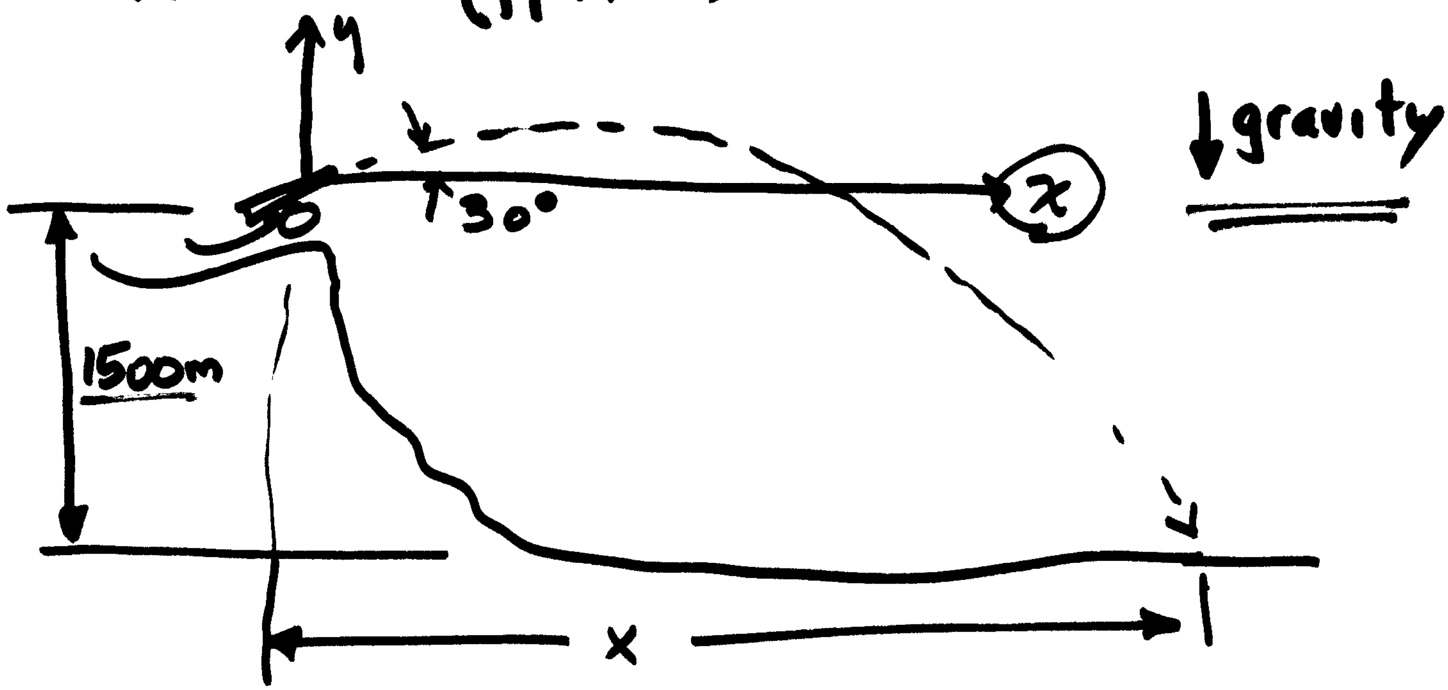
$$\underline{t' = \frac{4}{9} \text{ s}} \quad \begin{pmatrix}\text{time of} \\ \text{max } v\end{pmatrix} \text{(min?}$$

$$v_{max} = v(t') = 20 + 8\left(\frac{4}{9}\right) - 9\left(\frac{4}{9}\right)^2$$

$$\underline{= 21.8 \text{ m/s}}$$

(max. speed)

Problem 6 (pp 14-5)



Given: muzzle velocity 1000 m/s @ 30°.

Find: range, x

Solution: (Introd. cartesian Coord Syst.)

[Clue: is path known?
clue: is cylindrical geometry involved?]

Neglect air resistance

Acceleration is $\vec{a} = -g\hat{j}$

$$\vec{a} = -g\hat{j}$$

$$d\vec{v} = \int \vec{a}\, dt$$

9.81 m/s²

32.2 ft/s²

$$\vec{v} = \int_0^t \vec{a}(\hat{t})\,d\hat{t} + \vec{v}_0$$

$$\vec{v} = \vec{v}_0 + \int_0^t -g\hat{j}\,d\cancel{t} \quad (d\tau)$$

$$= 1000\,m/s\left[\cos 30^\circ\,\hat{i} + \sin 30^\circ\,\hat{j}\right] + (-g)\hat{j}\,t$$

$$\vec{v} = \left[1000\tfrac{m}{s}\cos\theta\right]\hat{i} + \left[1000\tfrac{m}{s}\sin\theta - 9.81\tfrac{m}{s^2}\,t\right]\hat{j}$$

(the velocity vector for all time $t > 0$)

$$\vec{r} = \int d\vec{r} = \int \vec{v}\,dt = \cancel{\vec{r}_0}^{\,0} + \int_0^t \vec{v}(\tau)\,d\tau$$

$$\vec{r} = \int_0^t 1000\tfrac{m}{s}\cos\theta\,\hat{i}\,d\tau + \int_0^t \left[1000\sin\theta - 9.81\tau\right]d\tau\,\hat{j}$$

~~15~~/16

$$\vec{r} = \left[1000\,\tfrac{m}{s}\cos\theta\, t\right]\hat{i} + \left[1000\,\tfrac{m}{s}\sin\theta\, t - \frac{9.81}{2}\tfrac{m}{s^2}\, t^2\right]\hat{j}$$

(the position vector for all times t)

"<u>range</u>" here means when $r_y = -1500$

set $r_y = \left[1000\sin\theta\, t - \frac{9.81}{2}t^2\right] = -1500$

$$\frac{9.81t^2}{2} - 1000\sin 30^\circ t - 1500 = 0$$

roots: $t = \begin{Bmatrix} -2.916\,s \\ +104.85\,s \end{Bmatrix}$

so,

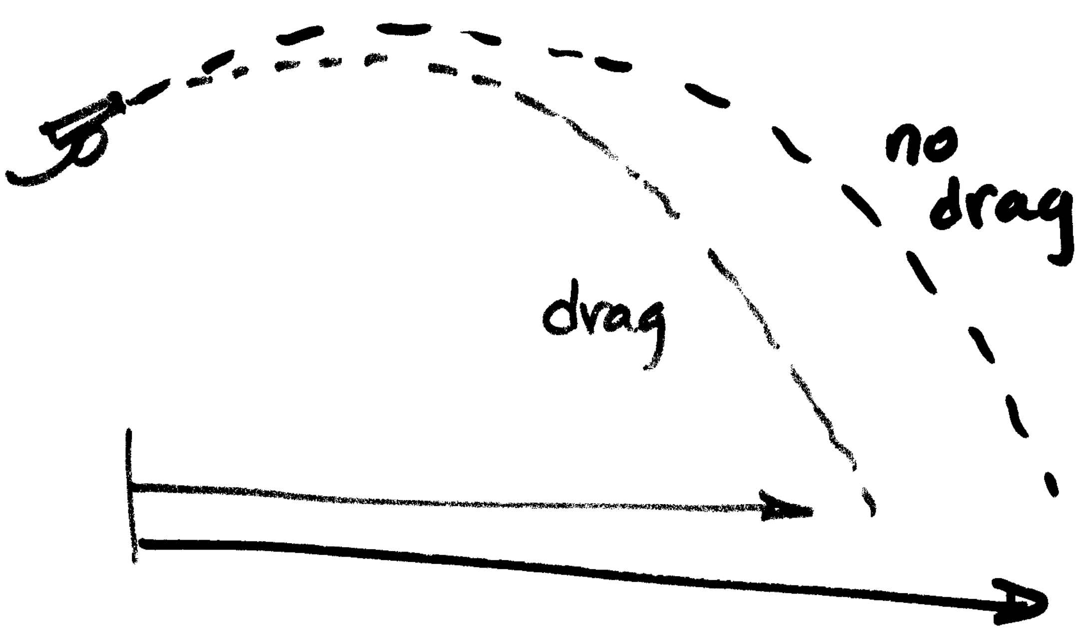
$$\text{range} = r_x(104.85\,s)$$

$$= (1000 \cos 30^\circ)(104.85\,s)$$

$$= 90\,803\text{ m}$$

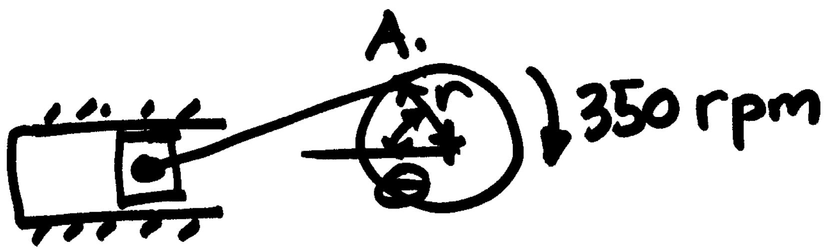
(What is the effect of air resistance?)

$$\vec{a} = -g\hat{j} - \frac{\vec{D}(\vec{v})}{m}$$



Problem #3 (pp 14-6)

Given: Recip. Pump, 350 rpm, r = 0.3 m



Find: velocity of point "A" when
$\theta = 35°$

Recall

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

clue (polar geometry suggests use of polar coordinates.)

$$\vec{v} = r\dot{\theta}\,\hat{e}_\theta \quad (\dot{r} = 0)$$

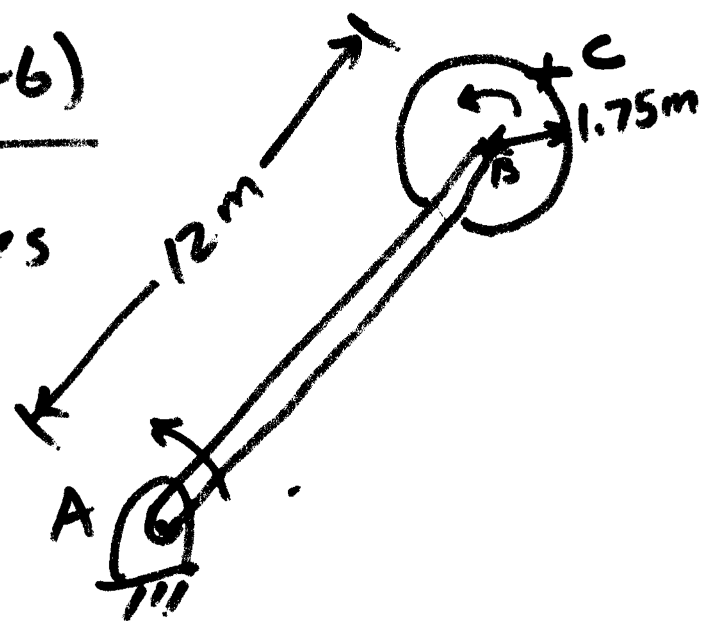
$r = 0.3\text{ m}$

$$\dot{\theta} = \left(\frac{350\text{ rev}}{\text{min}}\right)\left(\frac{2\pi\text{ rad}}{\text{rev}}\right)\left(\frac{1\text{min}}{60\text{s}}\right)$$

$= 36.65\text{ /s}$

$$v = r\dot{\theta}$$
$$= (0.3\,m)(36.65/s)$$
$$= 10.996\ m/s \simeq \underline{\underline{11.0\ m/s}} \quad (D)$$

Problem 5 (pp 14-6)



Given: Disk rotates ccw 60 rpm <u>relative to link</u>.

Link rotates 12 rpm ccw.

Find: Max velocity of any point on disk.

By inspection, point C will have max velocity.

$$v_c = v_{c/B} + v_B$$

~~(1.75m)(60 rpm)(2π) + (12)~~...

Plan Ahead!

$$\underline{v_c = v_{c/B} + v_B}$$

$$v_{c/B} = (1.75\,m)(60 + 12)^{rpm}\left(\frac{2\pi}{60}\right)$$

$$= \underline{13.195\ m/s}$$

$$v_B = (12\,m)(12\ rpm)\,\frac{2\pi}{60}$$

$$= \underline{15.080\ m/s}$$

$$v_c = 28.274\ m/s \qquad \underline{\underline{(B)}}$$

(Note: I believe solution of this problem in Manual is MIS-leading.)

Problem 10,11 (pp 14-7)

<u>Given</u> The position (radians) of a car traveling around a curve is given by $\theta(t) = t^3 - 2t^2 - 4t + 10$ (Ra

(<u>Hint</u>: polar coordinates may be advantageous, since $\underline{\theta(t)}$ is Known)

<u>10</u> Find: <u>angular velocity</u>* at $t = 3s$.

Soln: $\vec{r} = r\hat{e}_r$

$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

*(Interpretation: he wants $\dot{\theta}$)

$$\dot{\theta} = \frac{d\theta}{dt} = 3t^2 - 4t - 4$$

$$\underline{\dot{\theta}}\Big|_{t=3} = 3(9) - 12 - 4 = \underline{\underline{-11/\text{sec}}}$$

ans = $\underline{\underline{C}}$

Kinetics (Particles)

Definition:
The momentum of a particle is $\vec{p} = m\vec{v}$

Newton's Second Law:
(Empirical)

$$\Sigma \vec{F} = \frac{d}{dt}(\vec{p}) = \frac{d}{dt} m\vec{v} = m\vec{a}$$

(if $\frac{dm}{dt} = 0$)

UNITS

	L	T	F	M	
US	ft	sec	Lb	(slug)	(EES)
SI	m	s	(N)	kg	

(*) = (Derived units)

These are "consistent" units.
That is

$$F = MA$$

$$1\,\text{Lb} = (1\,\text{slug})(1\,\text{ft}/\text{sec}^2)$$

and $$1\,\text{N} = (1\,\text{Kg})(1\,\text{m}/\text{s}^2)$$

Note, there is no "g_c" required with consistent units, as is implied pp 15-1, eqn 15.1b.

Some write Lb_f and Lb_m to distinguish between F and M units. In my notation Lb means F. The mass that has a weight of 1 Lb is 1 Lb_m.

$$1\,\text{slug} = 32.2\ Lb_m$$

then $$W = mg$$

~~1 slug~~ ~~1 lb~~ (Plan ahead!!

In the "reference handook" another US system of units is used:

	L	T	F	M	
US	ft	sec	Lb_f	Lb_m	(USCS)

...this is <u>not</u> a consistent system of units. That is

$$F \neq MA$$

we must write $\Sigma \vec{F} = \frac{m}{g_c} \vec{a}$

where $g_c = \cancel{32.12} \cancel{\frac{Lb_f \cdot sec^2/ft}{Lbm}}$

(a "gravitational constant") $g_c = 32.18 \dfrac{Lbm}{Lb_f \cdot sec^2/ft}$

whenever this system is used, mass m must be replaced by $\frac{m}{g_c}$ in the equation of mechanics.

SUMMARY

UNITS

SYSTEM	L	T	M	F	g_c
SI	m	s	kg	(N)	1 (consistent)
US (EES)	ft	sec	(slug)	Lb	1 (")
US (USCS)	ft	sec	(Lb_m)	Lb_f	$\frac{32.18\ Lb_m \cdot ft}{Lb_f \cdot sec^2}$ (NOT consistent)

Weight:

The weight of a mass m in a gravitational field g is

$$W = mg$$

where

$$\left\{ \begin{array}{l} g = 9.81\ m/s^2 \\ \quad = 32.18\ ft/sec^2 \end{array} \right.$$

Examples:

The weight of 1 slug is

$$W = (1\ slug)(32.2\ ft/sec^2) = \underline{\underline{32.2\ lb}}$$

The weight of 1 kg is

$$W = (1\ kg)(9.81\ m/s^2) = \underline{\underline{9.81\ N}}$$

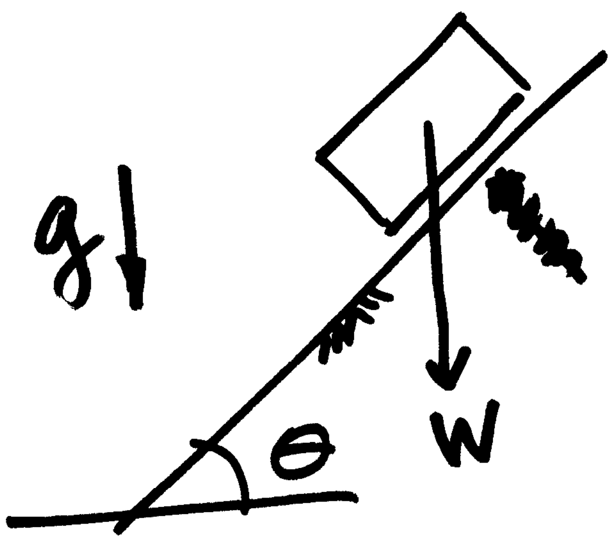
The weight of 1 Lbm is

$$W = \left(\frac{1\ Lbm}{g_c}\right)(32.2\ ft/sec^2) = \underline{\underline{1\ Lb_f}}$$

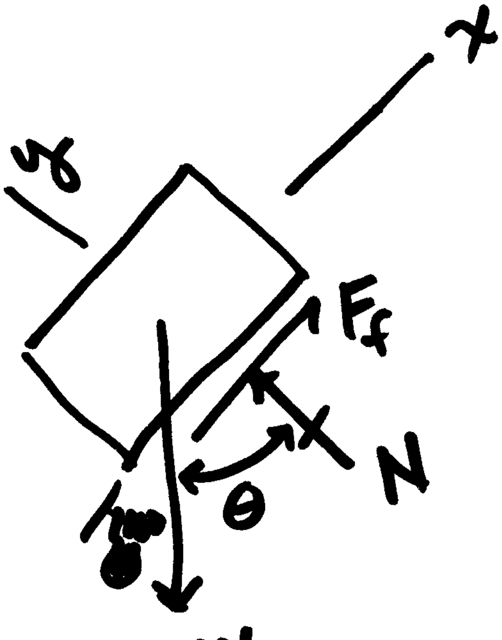
Friction

Coulomb friction $F_f \leq \mu N$

$(0 < \mu < 1)$



FBD



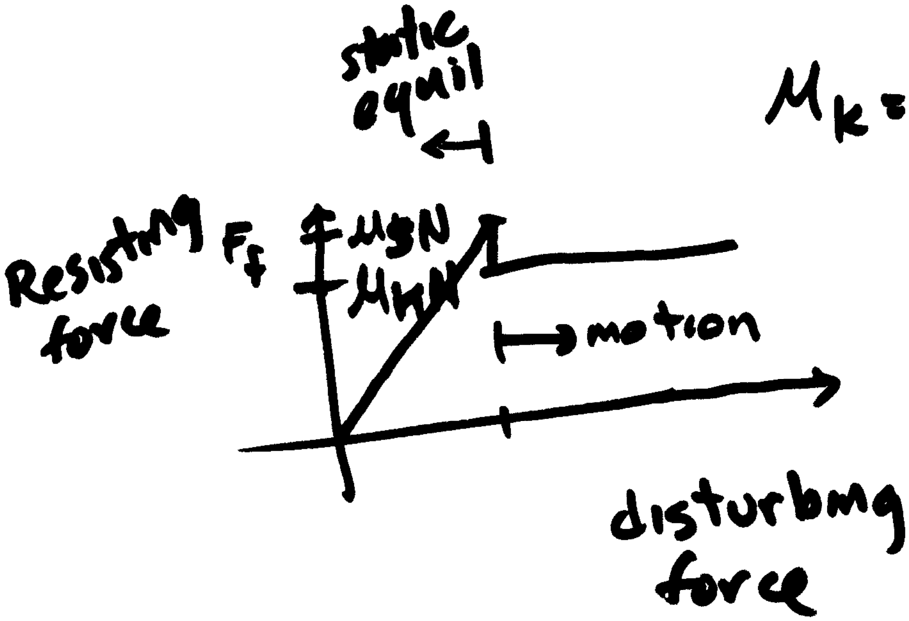
$\Sigma F_y = 0$ (rectilinear motion along x axis) $\Rightarrow a_y = 0$

$$N - W\cos\theta = 0$$
$$N = W\cos\theta$$

$$\underline{F_f \le \mu N}$$

we introduce μ_s = static coef. friction

μ_k = Kinetic coef. of friction



Tools:

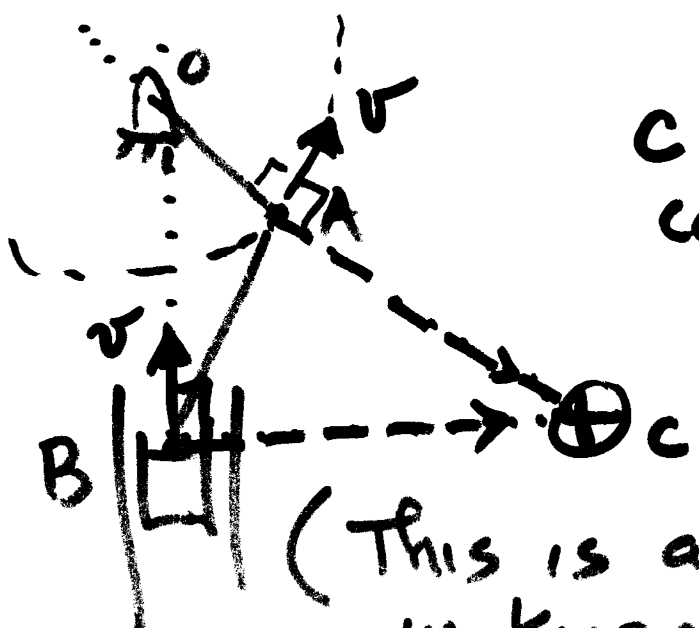
Free Body Diagram (FBD)

A F.B.D. shows all $\overrightarrow{\text{forces}}$ that act on an isolated body. Newton's Second law ($\sum \vec{F} = m\vec{a}$) is applied to the F.B.D.

Instantaneous Center

... that point, about which a body is (instantaneously) rotating.

Located by finding point of intersection of lines perpendicular to two velocity vectors at two points on the body



C is I.C. for connecting rod AB.

(This is a useful "tool" in Kinematics)

Back to Kinetics

From $\sum \vec{F} = m\vec{a}$

or $\vec{a} = \frac{1}{m}\sum \vec{F}$

given the forces $\sum \vec{F}$, we can determine the acceleration $\vec{a}$.

From Kinematics we can obtain: $\vec{v}(t)$

$\vec{r}(t)$

In the various coordinate systems we introduced:

Cartesian

$$\begin{aligned} \Sigma F_x &= m a_x \\ \Sigma F_y &= m a_x \\ \Sigma F_z &= m a_z \end{aligned}$$

then $\underline{v_x(t)} = v_x(0) + \int_0^t a(\tau)\, d\tau$

etc.

Polar

$$\begin{aligned} \Sigma F_r &= m a_r \\ \Sigma F_\theta &= m a_\theta \end{aligned}$$

then

$$\begin{aligned} (\ddot{r} - r\dot{\theta}^2) &= a_r \\ (r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= a_\theta \end{aligned}$$

Path (normal & tangential) Coord's.

$$\sum F_n = m\,a_n$$

$$\sum F_t = m\,a_t$$

$$\frac{v^2}{r} = a_n$$

$$\dot{v} = a_t$$

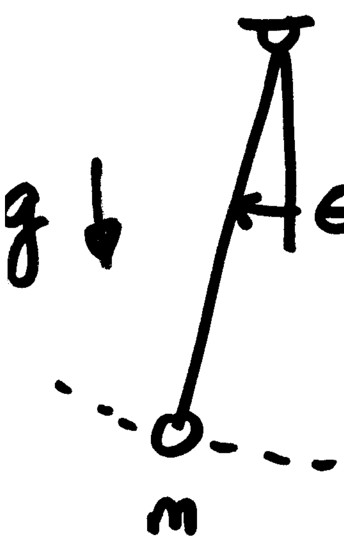
Example

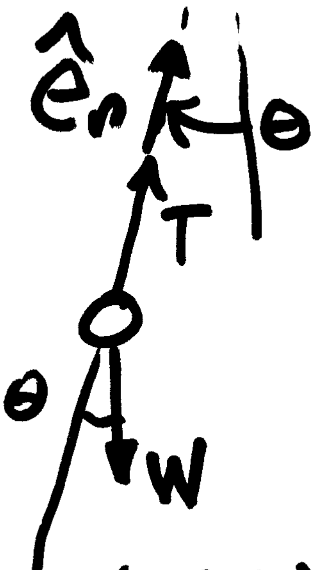
Problem 1. (pp 16-5)

Given: a 2kg mass swings in a vertical plane at the end of a 2m cord

~~Find:~~ The magnitude of tangential velocity is 1 m/s at $\theta = 30^\circ$

Find: Tension in the cord





(F.B.D.)

Kinematics:

$$a_n = v^2/\rho$$

(normal-tangential coords are suggested by the problem)

(Continued)

$$\sum \vec{F} = m\vec{a}$$

$$(\underline{\underline{n}})\quad T - W\cos\theta = m\frac{v^2}{\rho}$$

$$T = W\cos\underline{\theta} + m\frac{v^2}{\rho}$$

$$= \overbrace{(2\,kg)(9.81\,m/s^2)}\,\underline{\cos 30^\circ}$$

$$+ (2\,kg)\,\frac{\left(\underline{1\,m/s}\right)^2}{\underline{2m}}$$

$$T = 16.991 + 1. = 17.99\,\underline{\underline{N}}$$

$$= \underline{\underline{18.0\,N}}$$

Example

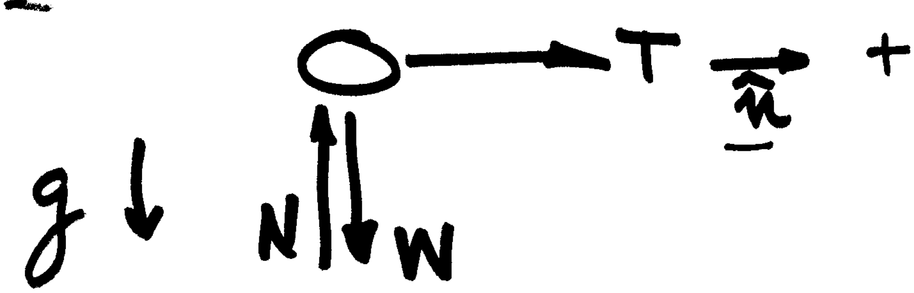
Problem 2 (pp 16-6)

Given: 2kg mass swings in horizontal circle of radius 1.5m by a taut cord with Tension T = 100N.

Find: Angular momentum of the mass (about the center of the circle).

$\vec{r} \times m\vec{v}$

Not in "Handbook"



(note motion is in horizontal plane, $\underline{N = W}$)

$$\Sigma F_n = m a_n$$

$$T = m\frac{v^2}{\rho}$$

$$v^2 = \frac{T\rho}{m}$$

$$v = \sqrt{\frac{100N\ 1.5m}{2kg}}$$

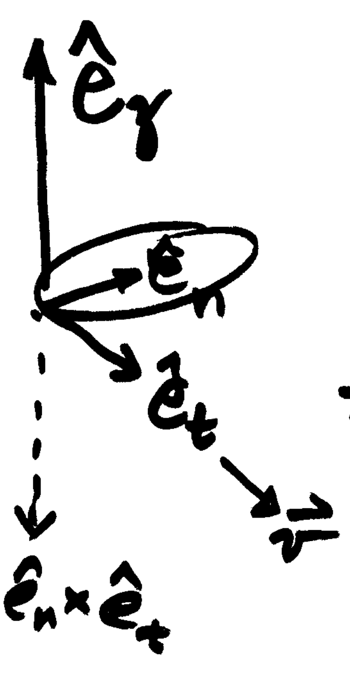
$$v = \sqrt{75} = \underline{\underline{8.66\frac{m}{s}}}$$

(continued)

Recall $\vec{p} = m\vec{v}$ is linear momentum

and $\vec{h} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$ is angular momentum

(here $\vec{r}$ is from point where angular momentum is to be calculated to point where mass m is located.)



$$\vec{p} = m\vec{v} = (2\,\mathrm{kg})(8.66\,\tfrac{m}{s})\,\hat{e}_t$$

$$\vec{r} = -1.5\,m\,\hat{e}_n$$

$$\vec{r} \times \vec{p} = (-1.5\,m)(2\,\mathrm{kg})(8.66\,\tfrac{m}{s})\,(-\hat{e}_z)$$

$$= 25.98\ \mathrm{kg\,m^2/s}$$

$$= 26.0\ \frac{(N)(m\cdot s)}{(N\cdot s)(m)}$$

Rigid Body Dynamics (2-D)

$$\sum \vec{F} = m\vec{a}_G$$

where $\vec{a}_G$ denotes the acceleration of the mass center, G, of the rigid body.

also $\sum M_G = I_G \alpha$

(Alternatively, you can write

$$\sum M_o = I_o \alpha$$

where point "o" is a pinned point, which has no acceleration.

I = mass moment of inertia

$$= \int_{vol} r^2 \, dm = \int_{vol} r^2 \rho \, dvol$$

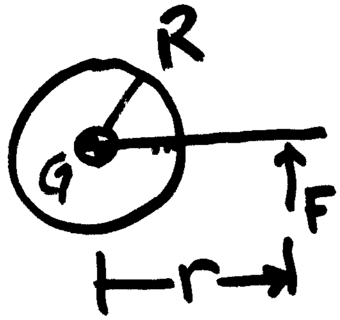
See tables <u>pp 28-29</u> in "Handbook"

M = moment of all forces on F.B.D. about point in question (G, "O")

α = angular acceleration ($\text{rad}/\text{sec}^2 = 1/s^2$)

Example (8, pp 16-7)

Given: Thin disk, radius 30 cm, mass 2 kg, with constant, tangential force, F = 10 N at unknown arm, r(t)



Given $\alpha = 3t\ /sec^2$.

Find: the unknown arm r(t) at t = 12 sec.

Soln.: at $t = 12s$

$$\alpha = 3t = 36\ /sec^2$$

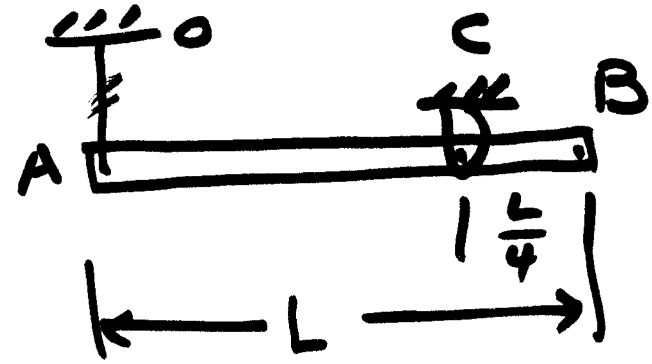
$$\Sigma M_G = I_G \alpha$$

$$F r = \left(\frac{mR^2}{2}\right) \alpha$$

$$r = \left(\frac{mR^2}{2}\right) \frac{\alpha}{F} = \frac{(2kg)(0.30m)^2}{2 \cdot 10N} \cdot 36./sec^2$$

$r = 0.324\ m$ (A)

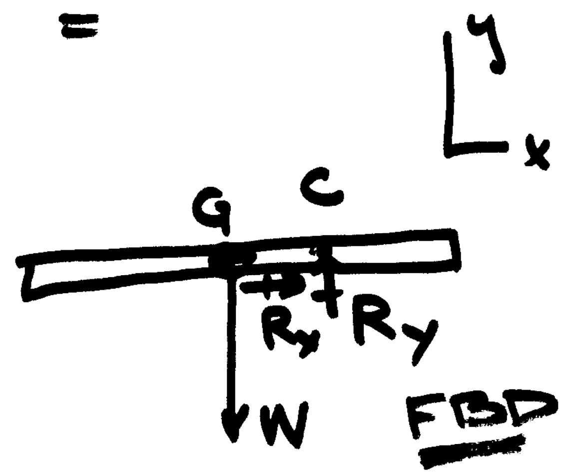
Example
(Prob. 13, pp 16-7)



Given: Uniform rod AB, Pinned at C. String OA is Cut.

Find: acceleration of $\underline{\underline{B}}$

Solution:



$$\underline{\Sigma M_C = I_C \alpha}$$

(Parallel Axis Theorem)

$$\underline{I_C} = \underline{I_G} + \underline{d^2 m}$$

where $d = L/4$ (dist. from C – G)

$$I_C = \frac{mL^2}{12} + m\left(\frac{L}{4}\right)^2 = \frac{mL^2}{48}(4+3) = \frac{7mL^2}{48}$$

$$\circlearrowleft M_C = \frac{WL}{4}$$

$$\alpha = \frac{M_C}{I_C} = \frac{mgL/4}{7mL^2/48} = \frac{12g}{7L}$$

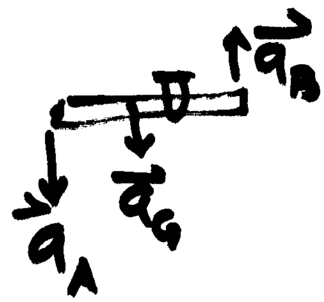
(cont.)

By Kinematics

$$\vec{a}_B = \cancel{\vec{a}_c}^{\,0} + \vec{\underline{\alpha}} \times \vec{r}_{B/c} + \cancel{\vec{\omega}\times\vec{\omega}\times\vec{r}}^{\,0}$$

$$\underline{a_B = \alpha L/4}$$

$$\underline{\underline{a_B = \frac{3g}{7}}} \quad (\underline{\underline{c}})$$



In addition to writing and solving Newton's 2nd law to obtain the acceleration, etc... the methods

work/Energy

<u>An</u>d

Impulse/Momentum

...are helpful.

(<u>Clue</u>: when asked for the "final" velocity, this indicates these two methods might be useful).

Work - Energy Method

$$\mathcal{U}_{1-2} = KE_2 - KE_1$$

...or, for a system, (say a rigid body F.B.D.) the work done on the system between configuration 1 and config 2 ($\mathcal{U}_{1-2}$) equals the change in Kinetic Energy

$$KE = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$$

(for a particle, $KE = \frac{1}{2} m v^2$)

~~The work can someti~~

Potential Energy

The work can sometimes be computed by the change in "Potential energy"

$$PE = wh = mgh$$

... is the PE due to gravity, relative to the reference datum from which h is measured.

$$PE = kx^2/2$$

... is the potential energy of an elastic spring (k) stretch (or compressed) x

then: $U_{1-2} = -(wh_2 - wh_1) = -w(h_2 - h_1$

or $U_{1-2} = -\left(\frac{kx_2^2}{2} - \frac{kx_1^2}{2}\right)$

Conservation of mechanical energy

When no work is done on a system (except by changes in potential) we can write

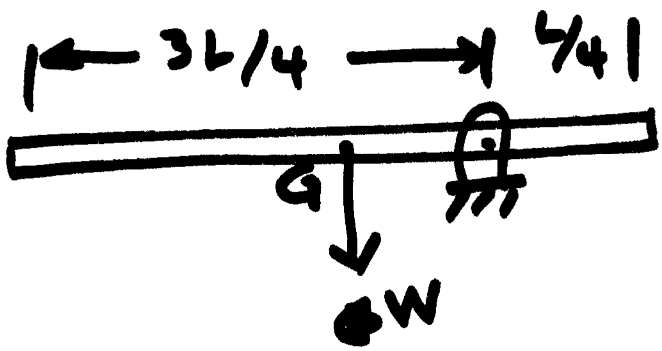
$$\mathcal{U}_{1\text{-}2} = -(PE_2 - PE_1) = KE_2 - KE_1$$

or $$\underline{PE_2 + KE_2 = PE_1 + KE_1} \checkmark$$

(Note: this is not always the case. For example, when frictional work is done, this special case does not apply)

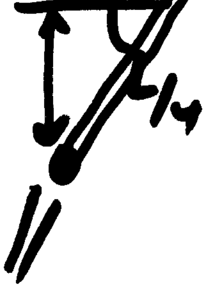
Example

The pinned beam shown falls 45°. What is the angular velocity ω at that time. (starts from rest)



(the "hint" observed is the need to find final velocities $\Rightarrow$ use work/energy)

Soln: at 45°, the beam has fallen $\frac{L}{4}\overset{\sin}{\cancel{\cos}}45° = \frac{L}{4\sqrt{2}}$



The work done (by gravity) is $U_{1-2} = \frac{WL}{4\sqrt{2}}$ (positive)

The work/energy says:

$$\mathcal{U}_{1-2} = KE_2 - KE_1$$

But $KE_1 = 0$ (starts from rest)

$$\mathcal{U}_{1-2} = \left(\frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2\right)_2$$

we can relate v_G and ω by kinematics

$$v_G = \frac{L}{4}\omega$$

$$\mathcal{U}_{1-2} = \frac{WL}{4\sqrt{2}} = \frac{1}{2} m \left(\frac{L}{4}\omega\right)^2 + \frac{1}{2}\left(\frac{mL^2}{12}\right)\omega^2$$

$$\frac{mgL}{4\sqrt{2}} = \frac{mL^2\omega^2}{2}\left(\frac{1}{16} + \frac{1}{12}\right)$$

$$\frac{mgL}{4\sqrt{2}} = \frac{mL^2\omega^2}{2}\left(\frac{3}{48} + \frac{4}{48}\right)$$

$$\omega^2 = \frac{2}{mL^2}\left(\frac{7}{48}\right)\frac{mgL}{4\sqrt{2}}$$

$$= \frac{7}{96\sqrt{2}}\, g/L \quad \left[\frac{1}{T^2}\right]$$

$$\underline{\underline{\omega = \sqrt{\frac{7}{96\sqrt{2}}\, g/L}}} \quad \left[\frac{1}{T}\right]$$ (units = 1/sec)

<u>Note</u>: If we knew $KE = \frac{1}{2} I_c \omega^2$ where point c is a pinned point. (<u>Not in "Handbook"</u>)

$$KE = \frac{1}{2}\left(\frac{7mL^2}{48}\right)\omega^2$$

... work is easier.

Example

A wheel (disk, mass m) comes off a vehicle traveling with speed v_0 on a horizontal plane.

How high will the wheel roll up the 1% incline?

$$\mathcal{U}_{1-2} = KE_2 - KE_1$$

Rolls w/o slip

$KE_2 = 0$ (when wheel is at top of travel)

$$KE_1 = \frac{1}{2} m v_0^2 + \frac{1}{2} I_G \omega^2$$

But $\omega = v_0 / r$. (Kinematics)

$$\mathcal{U}_{1-2} = -Wh$$

$$KE_2 \to 0$$

$$-Wh = -\left(\tfrac{1}{2} m v_0^2 + \tfrac{1}{2} I_G \left(\tfrac{v_0}{r}\right)^2\right)$$

$$mgh = \tfrac{1}{2} m v_0^2 + \tfrac{1}{2}\left(\tfrac{mr^2}{2}\right)\left(\tfrac{v_0}{r}\right)^2$$

$$mgh = \tfrac{1}{2} m \left(v_0^2 + \tfrac{v_0^2}{2}\right)$$

$$mgh = \tfrac{3}{4} m v_0^2$$

$$\underline{\underline{h = \tfrac{3}{4}\,\tfrac{v_0^2}{g}}} \qquad \underline{\underline{\frac{(m/s)^2}{(m/s^2)} = (m)}}$$

Given the friction force F
How far will the car skid,
if $W_{\cancel{car}\,T} = W_1 + 3W_{wheel}$?

$$\underline{\mathcal{U}_{1-2}} = \underline{K}E_2 - \underline{K}E_1$$

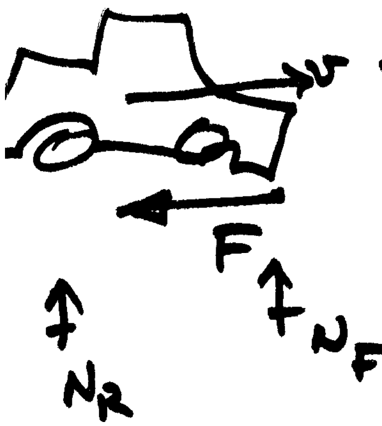
$$\underline{K}E_2 = 0 \quad \text{(again)}$$

$$KE_1 = \overbrace{\frac{1}{2}\frac{W_T}{g}}^{W+3W_w} v_0^2 + 3\left(\frac{1}{2} I_G \omega^2\right)$$

$$\omega = \frac{v_0}{r} \quad \text{(kinematics)}$$

$$KE_1 = \frac{1}{2}\frac{W_T}{g} v_0^2 + \frac{3}{2}\left(\frac{m_w r^2}{12}\right)\frac{v_0^2}{r^2}$$

$$KE_1 = \underline{\left(\frac{1}{2}\frac{W_T}{g} v_0^2 + \frac{3}{24}\frac{W_w}{g} v_0^2\right)}$$



$$U_{1-2} = -F d$$

(F = Frictional force
d = distance travelled)

(U_{12} is negative, since F is in opposite direction to travel.)

$$-Fd = -\frac{\left(\frac{1}{2} W_T + \frac{3}{24} W_w\right) v_0^2}{g}$$

$$\boxed{d = \frac{W_T + 3/12\, W_w}{2gF} v_0^2}$$

Impulse-Momentum

$$\vec{\mathcal{I}} = \vec{p}_2 - \vec{p}_1$$

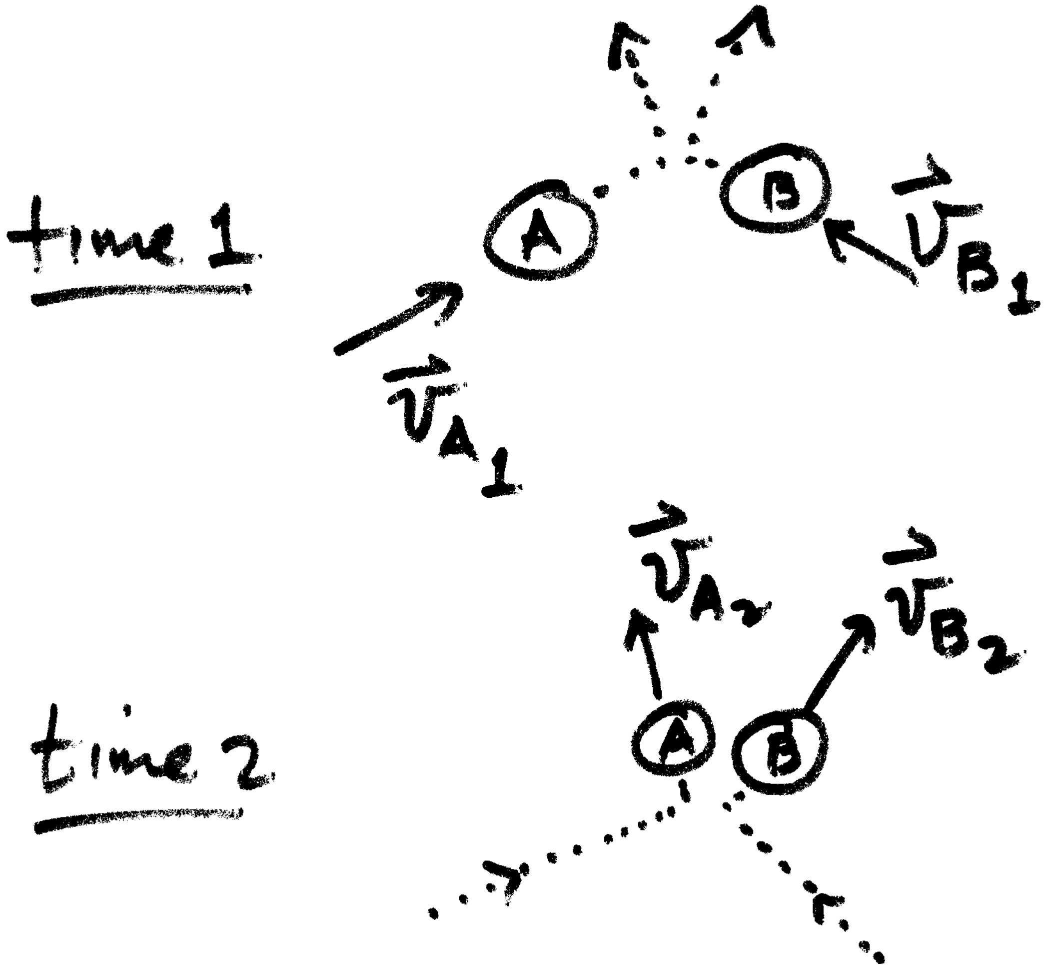
or: the $\overrightarrow{\text{impulse}}$ $\left(\vec{\mathcal{I}} = \int \vec{F} dt\right)$ $[F \cdot T]$

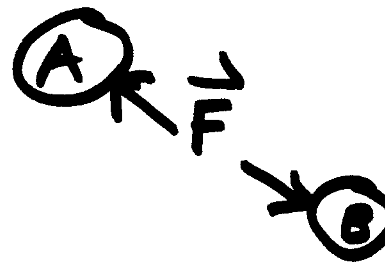
equals the change in linear momentum $\underline{\vec{p} = m\vec{v}_G}$

This is useful when...

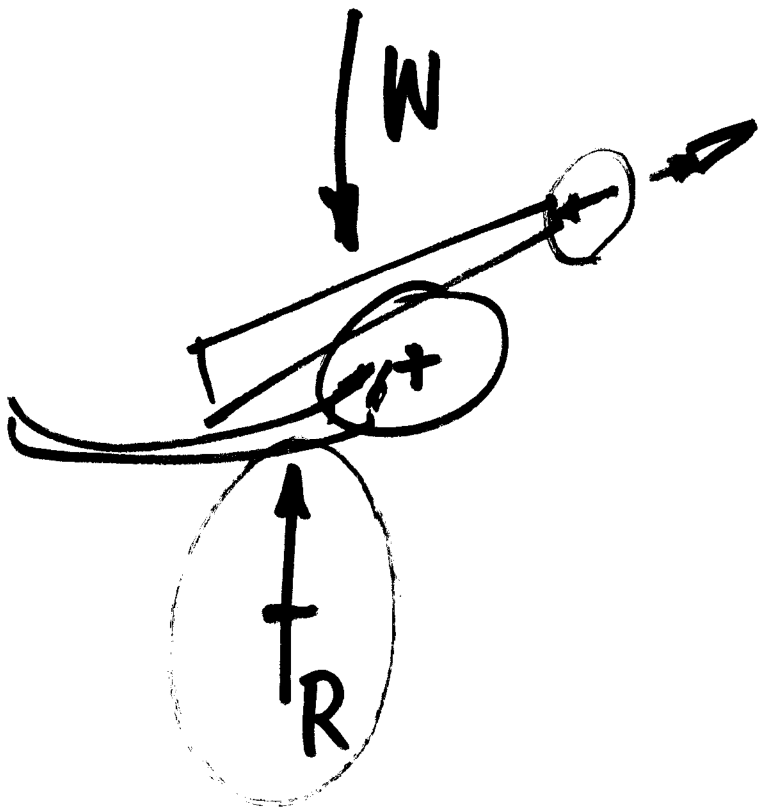
* • impacts occur
- force is a known function of $\underline{\underline{\text{time}}}$ (rather than $\underline{\text{position}}$) thus $\int \underline{\vec{F} dt}$ is known

In impact between 2 particles:





... the only forces that act are <u>internal</u> (between particles A & B) to the <u>system of A & B</u>.


W
R

thus $\int \vec{F}\,dt = 0$ (for A+B)

So, for the system of A+B,

we can say

$$0 = \left(m_A\vec{v}_{A_2} + m_B\vec{v}_{B_2}\right) - \left(m_A\vec{v}_{A_1} + m_B\vec{v}_{B_1}\right)$$

or $\vec{P}_2 = \vec{P}_1$ (linear momentum of the system is conserved, or constant)

This is true whether the particles are perfectly elastic spheres (ie billiard balls) or blobs of putty that stick together.

We describe the nature of the impact interaction with a "coefficient of restitution", e, defined as below:

In the direction normal to impact:

$$v_{rel_2} = -e \, v_{rel_1}$$

or, the relative velocity (normal) after impact is related to that before impact by $-e$.

$e = 1$ elastic

$e = 0$ perfectly plastic (adheres together)

$0 < e < 1$

Example

A golf ball bouncing on a concrete surface has $\underline{e = 0.9}$

What is the height of rebound when dropped from height h?

1. Use work/energy to determine velocity before impact
2. use impulse/momentum (really, just defn. of e) to obtain velocity after impact
3. use work/energy to get height of rebound.

1. $\rightarrow$ $v_1 = 2gh$ (steps skipped)

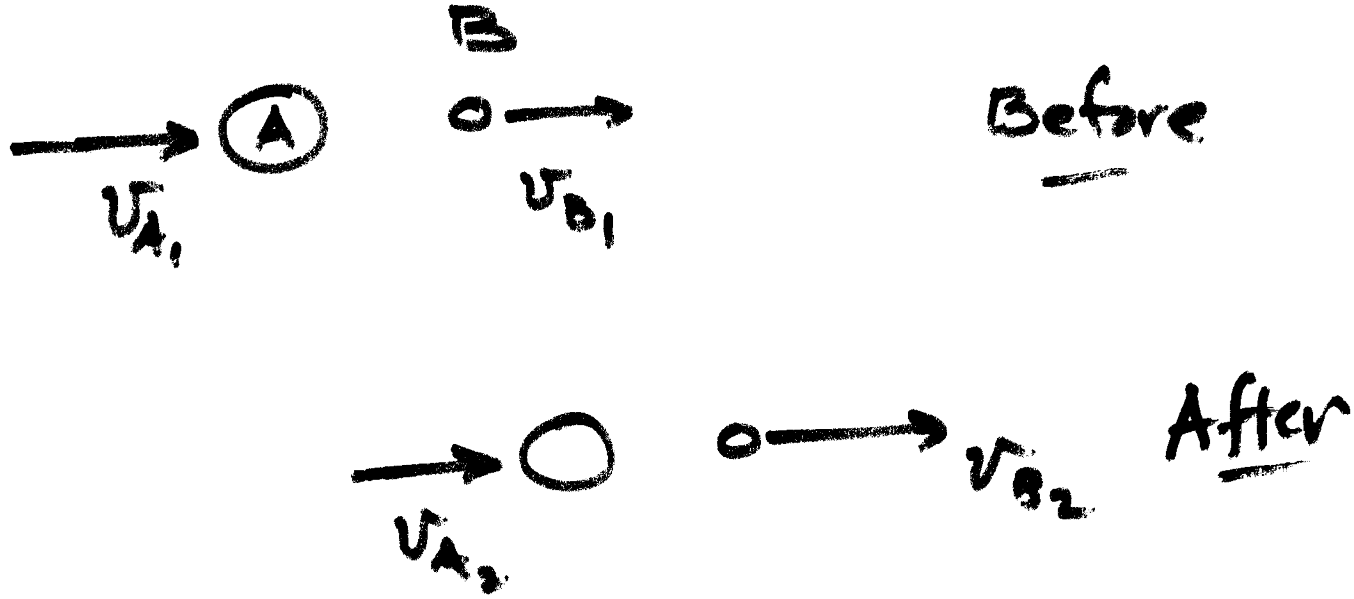
2. $\rightarrow$ $v_2 = -e v_1 = e(2gh)$ (upward)

3. $\rightarrow$ $h_2 = \dfrac{v_2}{2g} = \dfrac{e\,2gh}{2g} = \underline{\underline{eh}}$

$\therefore$ rebound height is 90% initial height.

Example

(124 slug) A 4000 Lb truck travelling at 44 ft/sec strikes a 2000 Lb auto travelling 20 ft/sec in same direction. Find velocity of the cars after impact assuming $\underline{\underline{e = 0.}}$[illegible]



For (A + B), only impulse is internal, (conserved)

so $\vec{P}_2 = \vec{P}_1$

$$m_A v_{A_2} + m_B v_{B_2} = m_A v_{A_1} + m_B v_{B_1}$$

$$4000\, v_{A_2} + 2000\, v_{B_2} = (4000)(44) + (2000)(20)$$

(Continued)

This is one equation in v_{A_2} an v_{B_2}.

Note that:

$$(v_{B_2} - v_{A_2}) = e(v_{A_1} - v_{B_1})$$

$$= e(44 - 20)$$

$$= 0.2\ (24\ ft/sec)$$

$$\underline{v_{B_2} = v_{A_2} + 4.8\ ft/sec}$$

$$4000\, v_{A_2} + 2000\,(v_{A_2} + 4.8) = 216\,000$$

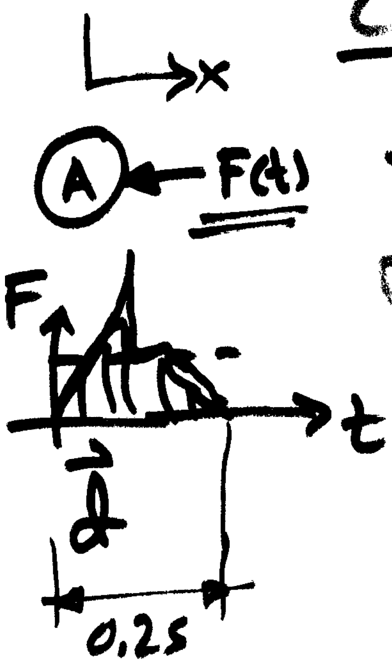
$$v_{A_2}\,(4000 + 2000) = 216000 - 9600$$

$$= 206{,}400$$

Soln

$$\begin{cases} v_{A_2} = 34.4\ ft/sec \\ v_{B_2} = 39.2\ ft/sec \end{cases}$$

We can also calculate the Impulse by looking at car A (or B).



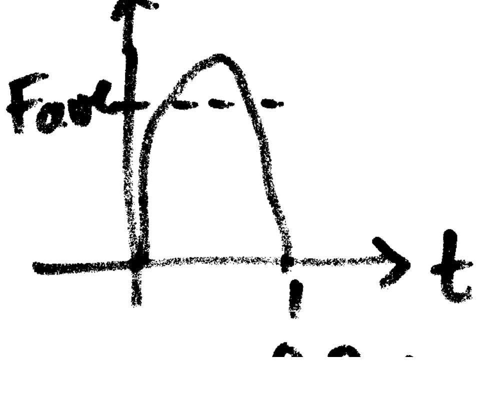
$$m = \frac{W}{g}$$

$$\vec{I} = \vec{P}_{A_2} - \vec{P}_{A_1}$$

$$= \frac{4000\,Lb}{32.2\,ft/sec^2}\left[34.4 - 44\right]\hat{i}$$

$$\vec{I} = -1193 \text{ Lb-sec } \hat{i}$$

If the duration of the crash is estimated to be 0.200 sec, then the forces (average) are



$$\vec{I} = \int \vec{F}\,dt = F_{ave}\,\Delta t$$

$$\vec{F}_{ave} = \frac{\vec{I}}{\Delta t}$$

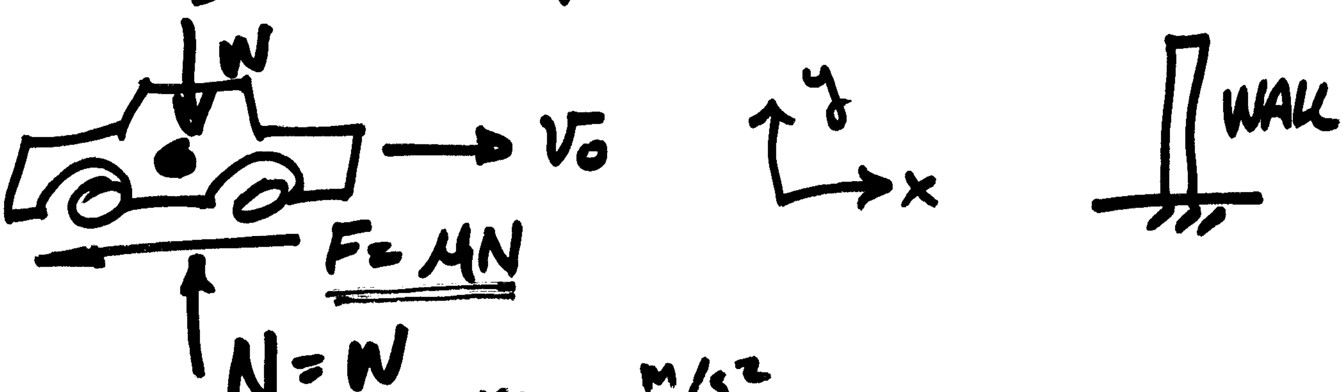
$$\vec{F}_{ave} = -5962\,\hat{i} \text{ Lb.}$$

Example (Prob. 16, pp 17-7)

Given: a 3500 kg car travelling at 65 Km/hr skids ($\mu = 0.60$) for 3 s and hits a wall.

Find: Velocity of impact

Note the clue: Force of friction is applied for a known time, 3 s, not a known distance → so use impulse/momentum.



$$W = mg = \overset{kg}{(3500)}\overset{m/s^2}{(9.81)} = \underline{34{,}335\ N.}$$

$$N = W$$

$$F = \mu N = \underline{20{,}601\ N.}$$

$$\vec{J} = \int \vec{F}\,dt = (-20{,}601)(3)\hat{\imath} = -61{,}803\ N{\cdot}s\ \hat{\imath}$$

$$-(20{,}601N)(3s)\hat{\imath}$$

(<u>Continued</u>)

$$\vec{J} = \vec{p}_2 - \vec{p}_1 = m\vec{v}_2 - m\vec{v}_1$$

$\uparrow$ initial

$\uparrow$ just before impact

$$m\vec{v}_2 = \vec{J} + m\vec{v}_1$$

$$3500\,\vec{v}_2 = -61{,}803\hat{i} + m\left(\frac{65000}{3600}\,m/s\right)\hat{i}$$

$$\vec{v}_2 = -\frac{61803}{3500}\hat{i} + 18.06\,m/s\,\hat{i}$$

$$\vec{v}_2 = -17.65\hat{i} + 18.06\hat{i} = \underline{0.398\hat{i}}$$

$$\underline{v_2 = 0.4\ m/s}$$

$$\underline{= 1.43\ \frac{Km}{hr}}$$

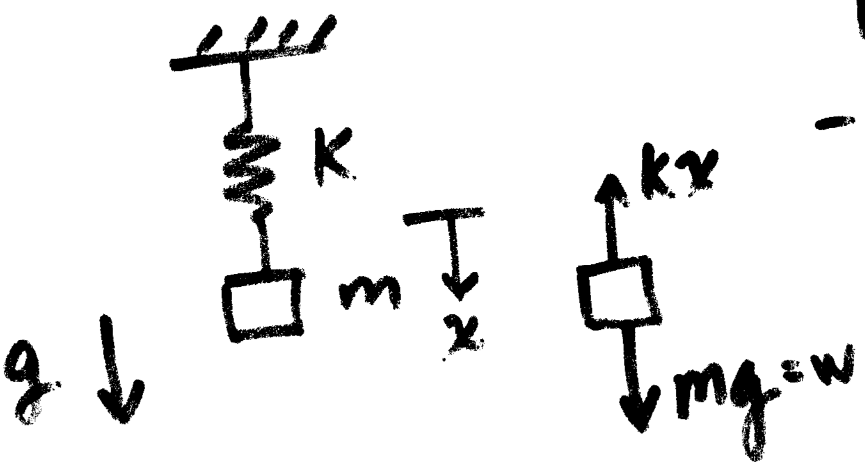
Free Vibration

Systems that lead to the differential equation

$$\boxed{\ddot{x} + \omega^2 x = 0}$$

are said to be "free vibration".

Example:



FBD

$$F = ma$$

$$-kx + mg = m\ddot{x}$$

$$m\ddot{x} + kx = W = mg$$

$$\boxed{\ddot{x} + \frac{k}{m} x = g}$$

$$\omega = \sqrt{\frac{K}{m}} \left[\frac{1}{s}\right]$$

$$\ddot{x}_h + \frac{k}{m} x_h = 0$$

(the homogeneous part of the DE)

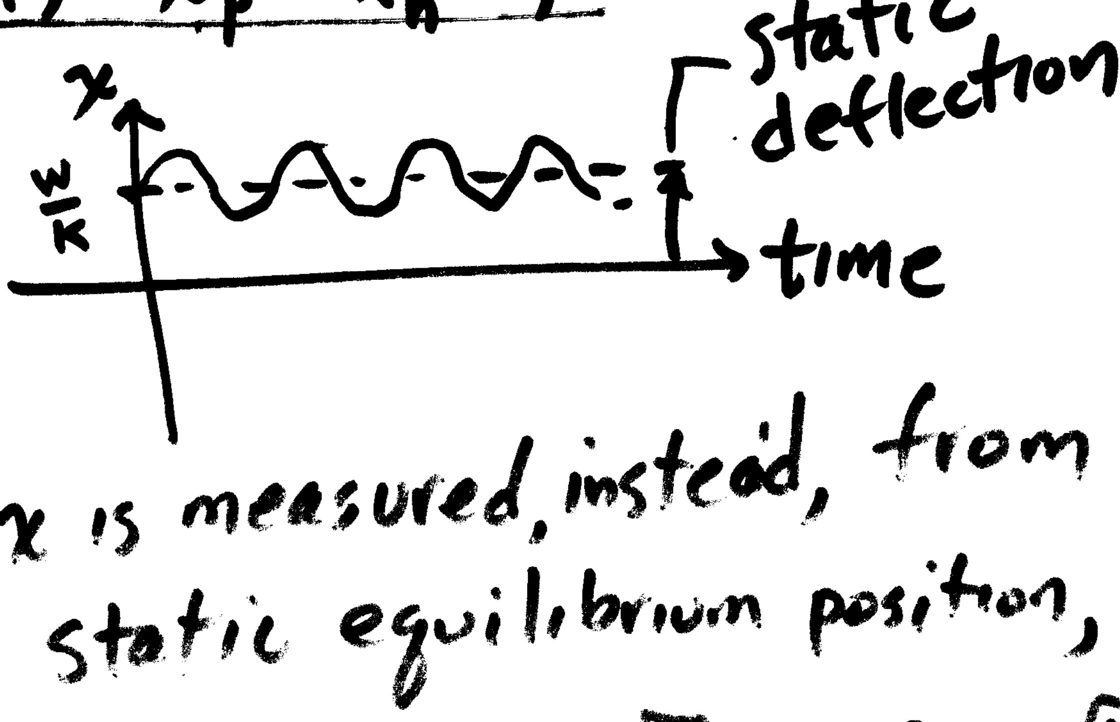
note $x_p = \frac{gm}{k} = \frac{W}{k}$ is a "particular" solution (the static deflection)

Solution to the homogeneous part are of the form

$$x_h(t) = C_1 \cos\sqrt{\frac{K}{m}}\,t + C_2 \sin\sqrt{\frac{K}{m}}\,t$$

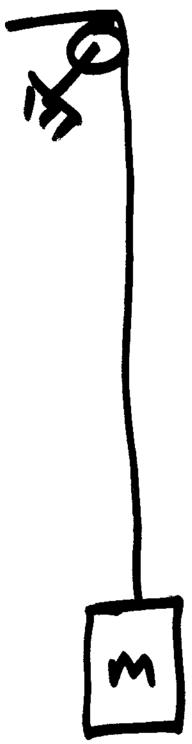
$\omega = \sqrt{K/m}$ is the natural frequency $[1/T]$

$$\underline{x(t) = x_p + x_h(t)}$$



...if x is measured, instead, from the static equilibrium position, then $x(t) = C_1 \cos\sqrt{K/m}\,t + C_2 \sin\sqrt{\frac{K}{m}}\,t$

Problem



A mine elevator has a cable $L = 1000$ m, supporting a cab of mass 4000 kg.

When the cab is initially installed on the cable, it is observed that the cable stretches 1.5 m.

Determine the expected natural frequency of vibration of the cab on the cable.

<u>Soln</u>: $W = (4000\,\text{Kg})(9.81) = 39{,}240\,N$

$$K = \left(\frac{1.5\,m}{39{,}240\,N}\right)^{-1} = 26{,}160\,N/m$$

$$\omega = \sqrt{K/m} = \sqrt{\frac{26{,}160}{4000}}$$

$$\underline{\underline{\omega = 2.56\ /sec}}$$

$$\text{Period} = \frac{2\pi}{\omega}\ \frac{(rad/cycle)}{(rad/sec)}$$

$$\underline{\underline{= 2.45\ sec}}$$

Overview

$$\sum \vec{F} = m\vec{a}_G$$

$$\sum M_G = I_G \alpha \quad (\text{or } \sum M_O = I_O \alpha)$$

$$U_{1-2} = KE_2 - KE_1 \quad * \quad KE = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$$

$$\vec{I} = \cancel{m\vec{v}_2}\,\vec{P}_2 - \cancel{m\vec{v}_1}\,\vec{P}_1 \qquad \vec{P} = \frac{1}{2} m \vec{v}_G^2$$

F.B.D.!

Clues about coordinate systems

Clues about when to use

N2L ←
W/E ←
I/M ←